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Metric Definition of μ-Homeomorphisms

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Dedicated to Fred and Lois Gehring

1. Introduction

The analytic definition of quasiconformality declares that a homeomorphism f between domains Ω and Ω' in $\mathbf{R}^n$, $n \geq 2$, is quasiconformal if $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \Omega')$ and there exists a constant K such that

$$|Df(x)|^n \leq KJ_f(x) \ \text{ a.e. in } \Omega.$$

Because the Jacobian of any homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$ is locally integrable, the regularity assumption on f in this definition can naturally be relaxed to $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$. There has been considerable interest recently in so-called μ-homeomorphisms that form a natural generalization of the concept of a quasiconformal mapping in dimension 2. To be more precise, we consider homeomorphisms $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$ such that

$$|Df(x)|^2 \leq K(x)J_f(x) \ \text{ a.e. in } \Omega \tag{1}$$

with $K(x) \geq 1$ and $\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega)$ for some $\lambda > 0$. A class of mappings equivalent to this was introduced by David in [2] and further studied in [17; 18]. David considered the Beltrami equation

$$\bar{\partial} f(z) = \mu(z)\partial f(z)$$

and essentially showed that a homeomorphic solution $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$ exists (in the planar case) when $|\mu(z)| \leq 1$ almost everywhere and

$$\exp\left(C\frac{1+|\mu(z)|}{1-|\mu(z)|}\right) \in L^1_{\mathrm{loc}}(\Omega)$$

for some $C > 0$; for this generality see [18]. These mappings in fact belong to $\bigcap_{p<2} W^{1,p}_{\mathrm{loc}}(\Omega, \Omega')$; they are differentiable a.e. and preserve the null sets for the 2-dimensional Lebesgue measure. These conclusions hold with 2 replaced by n in any dimension for mappings with an exponentially integrable distortion in the sense of (1); see [13; 14].

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