Michigan Math. J. 51 (2003)

Reinhardt Domains and Toric Models

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1. Introduction

This article is motivated by the construction of unitary representations of the torus, based on Kähler structures on strictly pseudoconvex Reinhardt domains. Let T be the compact n-torus. In the language of geometric quantization [7], the "classical" picture is a T-manifold X along with a T-invariant symplectic form ω, while the "quantum" picture is a unitary T-representation H. The process of transforming (X, ω) to $H = H_\omega$ is called *geometric quantization*. If H_ω contains every irreducible T-representation exactly once, it is called a *model*. This terminology is due originally to I. M. Gelfand and A. Zelevinski [4], who construct models of the classical groups. Since T is a torus, we also call it a *toric model*. A space where T acts naturally is the Reinhardt domain $X \subset \mathbf{C}^n$, since $(z_1, \dots, z_n) \in X$ implies that $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in X$. Consider the setting (X, ω), where ω is a T-invariant Kähler form on the Reinhardt domain X. The central issue of this article is: *When does (X, ω) provide a toric model H_ω?*

We shall describe the Kähler structures ω, construct H_ω, and show that the conditions for H_ω to be a toric model are closely related to the convergence of the integrals

$$\int_{x \in \Omega} e^{-F(x) + \lambda x}\, dV, \quad \lambda \in \mathbf{Z}^n.$$

Here $\Omega \subset \mathbf{R}^n$ is a strictly convex domain, $F \in C^\infty(\Omega)$ a strictly convex function, and dV the Lebesgue measure. This integral will be our major concern. We now outline our projects in better detail.

We restrict our consideration to strictly pseudoconvex Reinhardt domains X with free T-action. Free T-action implies that if $(z_1, \dots, z_n) \in X$ then $z_i \neq 0$ for all i. By the exponential map and normalization $2\pi \sim 1$, it follows that T and X have the following convenient descriptions:

$$T = \mathbf{R}^n / \mathbf{Z}^n, \qquad X = \left\{ x + \sqrt{-1}\, y : x \in \Omega,\ y \in T \right\}, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^n$ is a domain. Note that $X = \Omega + \sqrt{-1}\, T$. The T-action on X is given by addition on $y \in \mathbf{R}^n / \mathbf{Z}^n$ in (1.1). We shall see in Section 2 that strict pseudoconvexity of X leads to nice convexity properties of Ω. From now on, X, T, Ω

Received November 20, 2001. Revision received October 2, 2002.
This work is supported in part by the National Science Council of Taiwan.