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Homoclinic Orbits for Schrödinger Systems

Martin Schechter & Wenming Zou

1. Introduction

We consider the following Schrödinger system:

$$\begin{cases} \partial_t u - \Delta_x u + V(x)u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v + V(x)v = H_u(t,x,u,v) \end{cases} \quad \text{for } (t,x) \in \mathbf{R} \times \mathbf{R}^N, \tag{S}$$

where $V : \mathbf{R}^N \to \mathbf{R}$ and $H : \mathbf{R} \times \mathbf{R}^N \times \mathbf{R}^{2M} \to \mathbf{R}$ are periodic in t and x; $(u,v) \equiv (0,0) \in \mathbf{R}^{2M}$ is a stationary solution. Our purpose is to find a nonstationary solution $z = (u,v) : \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^{2M}$ of (S) satisfying $z(t,x) \to 0$ as $|t| + |x| \to \infty$. In this case, it is called the homoclinic orbit that is homoclinic to the stationary solution.

During the last ten years, the existence of homoclinic solutions has been studied by variational methods (see e.g. [AB1; AB2; CR; ScZ; SZ; WZ] and the references cited therein). Since there is no compactness of imbedding, the problem becomes very complicated. The difficulty also occurs when we consider (S). Before stating the main results, let us recall some well-known results related to (S). Brézis and Nirenberg [BrN] considered the system

$$\begin{cases} \partial_t u - \Delta_x u = -v^5 + f \\ -\partial_t v - \Delta_x v = u^3 + g \end{cases} \quad \text{for } (t,x) \in (0,T) \times \Omega$$

satisfying $u = v = 0$ on $(0,T) \times \partial\Omega$ and $u(0,x) = v(T,x) = 0$ in Ω. Here Ω is a bounded domain of $\mathbf{R}^N$ and $f, g \in L^\infty(\Omega)$. Using Schauder's fixed point theorem, they obtained a solution (u,v) with $u \in L^4((0,T) \times \Omega)$ and $v \in L^6((0,T) \times \Omega)$.

In [CFM], the authors studied the following problem:

$$\begin{cases} \partial_t u - \Delta_x u = |v|^{q-2} v \\ -\partial_t v - \Delta_x v = |u|^{p-2} u \end{cases} \quad \text{for } (t,x) \in (-T,T) \times \Omega,$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$ and $N/(N+2) < 1/p + 1/q < 1$. By the usual mountain pass theorem, they obtained at least one positive solution satisfying $u(t,\cdot)|_{\partial\Omega} = v(t,\cdot)|_{\partial\Omega} = 0$ for all $t \in (-T,T)$ and for $u(-T,\cdot) = u(T,\cdot)$ and $v(-T,\cdot) = v(T,\cdot)$.

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