Michigan Math. J. 51 (2003)

Infinitely Many Grand Orbits

Mark Comerford

1. Introduction

Sullivan's non-wandering theorem is one of the best-known and most fundamental results of classical rational iteration. We exhibit a counterexample which shows that a natural consequence of this theorem no longer holds if one is allowed to choose a different polynomial at each stage of the iterative process. The proof relies heavily on properties of the local dynamics near a parabolic fixed point.

We begin by considering a sequence of rational functions $\{R_n\}_{n=1}^{\infty} = \{R_1, R_2, R_3, \dots\}$ of some fixed degree $d \geq 2$. Let $Q_n(z)$ be the composition of the first n of these functions in the natural order; that is,

$$Q_n = R_n \circ R_{n-1} \circ \cdots \circ R_2 \circ R_1.$$

We will also be interested in the compositions

$$Q_{m,n} = R_n \circ R_{n-1} \circ \cdots \circ R_{m+2} \circ R_{m+1}.$$

Define the *Fatou set* $\mathcal{F}$ for such a sequence of rational functions as

$$\mathcal{F} = \{z \in \bar{\mathbb{C}} : \{Q_n\}_{n=1}^{\infty} \text{ is a normal family on some neighbourhood of } z\};$$

the *Julia set* $\mathcal{J}$ is then simply the complement of the Fatou set in $\bar{\mathbb{C}}$. Note that, if $\{R_n\}_{n=1}^{\infty}$ is a constant sequence $\{R, R, R, \dots\}$, then these definitions coincide with the standard ones. One of the reasons for this definition of Julia and Fatou sets is that we can formulate an analogue of the principle of complete invariance in standard rational iteration. In order to do this, we shall introduce the following terminology.

We start by fixing as before a sequence $\{R_n\}_{n=1}^{\infty} = \{R_1, R_2, R_3, \dots\}$ of rational functions of fixed degree $d \geq 2$. With this in mind, for any $n \geq 0$, let us define the nth *Julia set* $\mathcal{J}_n$ to be the Julia set for the sequence $\{R_{n+1}, R_{n+2}, R_{n+3}, \dots\}$ that we obtain from our original sequence simply by deleting the first n members. The nth *Fatou set* is similarly defined as the Fatou set for $\{R_{n+1}, R_{n+2}, R_{n+3}, \dots\}$. Note that, with these definitions, $\mathcal{J}_0 = \mathcal{J}$ and $\mathcal{F}_0 = \mathcal{F}$. We now state the principle of complete invariance for random iteration as follows.

Theorem 1.1. *For any $0 \leq m < n$ we have $Q_{m,n}(\mathcal{J}_m) = \mathcal{J}_n$ and $Q_{m,n}(\mathcal{F}_m) = \mathcal{F}_n$, with Fatou components of $\mathcal{F}_m$ being mapped surjectively onto those of $\mathcal{F}_n$ by $Q_{m,n}$.*

Received November 12, 2001. Revision received March 4, 2002.