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Affine Dimension: Measuring the Vestiges of Curvature

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1. Introduction

The purpose of this paper is to introduce a set function, which we call affine dimension, and then apply it to the study of convex curves in $\mathbb{R}^2$. Though we believe the geometric results obtained here—as well as questions concerning their higher-dimensional generalizations—to be intrinsically interesting, the original motivation for introducing affine dimension stems from its connection (as a natural necessary condition) to certain important problems in harmonic analysis. This connection is spelled out in Proposition 2 at the end of this section (see also Propositions 3 and 4 in Section 3). For earlier applications to harmonic analysis of special cases of the affine measures introduced here, see [1; 2; 7; 8].

The definition of affine dimension is largely analogous to the definition of Hausdorff dimension. But there is an important difference that, as we shall see, renders affine dimension sensitive to curvature. For example, the Hausdorff dimensions of a circle and a line segment are equal, but their affine dimensions will differ. The following observation is intended to motivate the definition of affine dimension: For $p \geq 2$, consider the smallest rectangle containing the portion of the curve (t, t^p) corresponding to $0 \leq t \leq \varepsilon$; that rectangle has measure on the order of ε^{1+p}, which tends to 0 as p increases—that is, as the curve (t, t^p) becomes "flatter". Thus, for $E \subseteq \mathbb{R}^n$ and $\alpha, \delta > 0$, we consider sums of the form $\sum |R_j|^{\alpha/n}$, where $E \subseteq \bigcup R_j$, each R_j is a rectangle of diameter $< \delta$, and $|R_j|$ is the Lebesgue measure of R_j. By analogy to the definition of Hausdorff measures, we define $A^\alpha_\delta(E)$ to be the infimum of such sums. Next we define

$$A^\alpha(E) \doteq \lim_{\delta \to 0^+} A^\alpha_\delta(E).$$

One sees in the usual way that A^α is an outer measure on $\mathbb{R}^n$ that restricts to a measure on the σ-algebra of Borel subsets of $\mathbb{R}^n$. We will refer to this measure as α-dimensional affine measure on $\mathbb{R}^n$. The equivalent definition with parallelepipeds instead of rectangles is clearly invariant under equiaffine transformations (as defined in [5], these are the affine mappings on $\mathbb{R}^n$ that preserve Lebesgue measure). Finally, we define $\dim_a(E)$ to be the infimum of $\{\alpha > 0 : A^\alpha(E) < \infty\}$. If $\dim(E)$ stands for the Hausdorff dimension of $E \subseteq \mathbb{R}^n$, then it is clear from the

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