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Examples Relating to the Crossing Number, Writhe, and Maximal Bridge Length of Knot Diagrams

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1. Introduction

The "Perko pair" knot $10_{161} = 10_{162}$ [R, p. 415; identification noted in second printing] is exceptional in at least two distinct ways. It first achieved its name and fame when Perko [P] discovered that the knot refutes a conjecture of Tait: it possesses two 10-crossing diagrams (the minimum possible) with distinct writhes. All previous knot tables of that depth had incorrectly listed the diagrams as representing distinct knots.

The same knot refutes a much shorter-lived conjecture concerning one of the link polynomials discovered in the 1980s. Let $Q(z)$ be the polynomial of Brandt, Lickorish, and Millett [BLM] and Ho [Ho]. Define an *overbridge* in a link diagram to be a consecutive sequence of overcrossing segments, and define an *underbridge* (the natural word "tunnel" has a different meaning in low-dimensional topology) to be a consecutive sequence of undercrossing segments. A *bridge* is the common term for both under- and overbridges. Define the *length* of a bridge to be the number of segments overcrossed or undercrossed. Kidwell [K] proved that, if a knot has a diagram with c crossings and its longest (over or under-) bridge has length d, then

$$\deg Q(z) \le c - d. \tag{a}$$

(Kidwell was thinking only of overbridges at the time; the dual underbridges were more recently pointed out to him by Stoimenow.) Kidwell asked (see [M2]) whether every knot or link has a diagram for which (a) is an equality, knowing that this was probably too much to hope for, since it would imply that the unknot is characterized as the unique knot with $Q(z) = 1$ (still an intractable open question). Equality in (a) can be achieved for alternating knots and for all knots with minimal crossing number below 10. The Perko pair, however, became a leading candidate to refute the conjecture. It has recently been demonstrated [SK] that relation (a) is a strict inequality for every diagram of the Perko pair.

Kauffman [Ka] soon generalized $Q(z)$ to his two-variable polynomial $F(a, z)$, and Thistlethwaite [T1] investigated its degree properties. He proved an analogous inequality to (a):

$$\deg_z F(a, z) \le c - d. \tag{b}$$

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