ON MANIFOLD-LIKE POLYHEDRA

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1. INTRODUCTION

If Π is a collection of spaces, a metric space X is said to be Π-like if for each $\varepsilon > 0$ there exists a map

$$f: X \to Y \in \Pi$$

that is onto and all of whose point inverses have diameter less than ε. Ganea [3] has given an example of an S^3-like space that is not a manifold, and Deleanu [2] has proved that every manifold-like polyhedron of dimension less than 4 is a manifold. This note gives an example, for each $n \geq 4$, of an n-dimensional, S^n-like polyhedron that is a generalized manifold but not a manifold.

2. CONSTRUCTION

By a theorem of Curtis [1] there exists, for each $n \geq 4$, a combinatorial n-manifold M with the properties

(1) M is contractible,

(2) $\pi_1(\partial M) \neq 0$ but M is a homology sphere,

(3) $M \times I = I^{n+1}$.

Let N be the suspension of ∂M. Then N can be written as $C(\partial M \times i) \cup \partial M \times I$ $(i = 0, 1)$, where $C(X)$ means the cone over X. Moreover, if $\varepsilon > 0$, we may take $C(\partial M \times i)$ to have diameter less than ε. Since M is contractible, there exists a map $h: C(M) \to M$ that is the identity on the base of the cone. Let

$$g = h \mid C(\partial M): C(\partial M) \to M.$$

We shall show that g is onto. Indeed, assume that this is not so, and let $x \in M - \mathrm{im}(g)$. Since g is the identity on ∂M, x is in the interior of M. Let U be the interior of a combinatorial n-cell containing x, and let it be small enough so that $\overline{U}$ misses $\mathrm{im}(g)$. Then $M - U$ is an orientable combinatorial manifold with boundary $\partial M \cup \partial\overline{U}$, and therefore the fundamental $(n - 1)$-cycles on ∂M and $\partial\overline{U}$ are homologous in $M - U$. From the homology exact sequence of the pair $(M, M - U)$ it is clear that the fundamental $(n - 1)$-cycle on $\partial\overline{U}$ generates $H_{n-1}(M - U) = Z$. Then the inclusion $i: \partial M \to M - U$ induces an homology isomorphism in dimension $n - 1$. Since g is the identity on ∂M, the diagram

$$\begin{tikzcd} H_{n-1}(\partial M) \arrow[rr, "i_*"] \arrow[dr] & & H_{n-1}(M - U) \\ & H_{n-1}(C(\partial M)) \arrow[ur, "g_*"'] & \end{tikzcd}$$

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