ASYMPTOTIC VALUES OF MEROMORPHIC FUNCTIONS

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1. INTRODUCTION

Let $\mathfrak{D}$ denote the unit disc $\{|z| < 1\}$, and let $\mathfrak{C}$ denote the unit circle $\{|z| = 1\}$. The purpose of this paper is to derive some results on asymptotic values of functions meromorphic in $\mathfrak{D}$. G. R. MacLane [12, p. 7] considered the classes $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{L}$ of functions that are nonconstant and holomorphic in $\mathfrak{D}$. $\mathscr{A}$ is the class of functions having asymptotic values at a dense set on $\mathfrak{C}$. $\mathscr{B}$ is the class of functions for which there exists a set of Jordan arcs Γ in $\mathfrak{D}$, with end points dense on $\mathfrak{C}$, such that on each Γ either $f \to \infty$ or f is bounded. The class $\mathscr{L}$ is defined as follows: $f \in \mathscr{L}$ if and only if each level set $\{z: |f(z)| = \lambda\}$ "ends at points" of $\mathfrak{C}$ (the precise definition will be found early in Section 3). MacLane proved that $\mathscr{A} = \mathscr{B} = \mathscr{L}$. We shall consider the corresponding classes $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{L}_m$ of meromorphic functions.

The classes $\mathscr{A}_m$, $\mathscr{B}_m$, and $\mathscr{L}_m$ are defined in Section 3. We prove that

$$\mathscr{A}_m \subset \mathscr{B}_m \quad \text{and} \quad \mathscr{L}_m \subset \mathscr{B}_m,$$

and we give examples showing that

$$\mathscr{B}_m \not\subset \mathscr{A}_m, \quad \mathscr{B}_m \not\subset \mathscr{L}_m, \quad \mathscr{A}_m \not\subset \mathscr{L}_m, \quad \mathscr{L}_m \not\subset \mathscr{A}_m.$$

Section 4 is concerned with the existence of asymptotic values on sets of positive measure. We prove (Theorem 5) that if $f \in \mathscr{A}_m$ and there exists a complex number a (possibly ∞) such that $N(r, a, f) = O(1)$, then on each subarc γ of $\mathfrak{C}$ on which f does not have the asymptotic value a, f has asymptotic values on a set of positive measure. Here $N(r, a, f)$ denotes the Nevanlinna counting function of f. Theorem 5 generalizes a theorem of MacLane [12, Theorem 11]. This result, together with Theorem 8, extends a theorem of Bagemihl [1, Theorem 1], which is a generalization of [4, Theorem 3].

In Section 5 we establish sufficient conditions for f to belong to $\mathscr{A}_m$. The fundamental condition (see Theorem 7) is as follows. If there exist a complex number a (possibly ∞) and a set Θ, dense on $[0, 2\pi]$, such that

$$\int_0^1 (1 - r)\log^+\left|\frac{1}{f(re^{i\theta}) - a}\right| dr < \infty \quad \text{and} \quad N(r, a, f) = O(1) \qquad (\theta \in \Theta,\ a \neq \infty),$$

then $f \in \mathscr{A}_m$. (If $a = \infty$, change $1/(f - a)$ to f.) A more restrictive condition is

$$\int_0^1 (1 - r)\,T(r)\,dr < \infty \qquad \text{and} \qquad N(r, a, f) = O(1),$$

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