THE CONJUGATE SPACE OF THE SPACE OF SEMIPERIODIC SEQUENCES

I. D. Berg

We are primarily concerned with the presentation of a construction for the conjugate space of the space of semiperiodic sequences. In the somewhat expository introduction, we consider the general question of construction of conjugate spaces of algebras of almost periodic functions on the semigroup of positive integers, and we motivate the construction of the particular conjugate spaces that we undertake in Sections 2 and 3.

1. INTRODUCTION

We let Z^+ denote the semigroup of positive integers, and Z the group of all integers.

If $\{x(n)\}$ is a complex sequence, we say that $\{x(n)\}$ is *periodic of period* p if $x(n+p) = x(n)$ for all $n \in Z^+$. We say that a sequence is *periodic* if there exists a $p \in Z^+$ such that the sequence has period p. We denote by P the space of periodic sequences, and by Q the closure of P in the supremum norm. We call Q the space of *semiperiodic sequences*, and it is largely with this space that we are concerned.

The space Q can be made into a Banach algebra with the obvious coordinatewise operations. As such, $Q = C(\bar{\omega})$, where the compact group $\bar{\omega}$ is the dual of the group of rationals modulo 1 in the discrete topology. That is, Q is the algebra of all continuous complex-valued functions on $\bar{\omega}$.

We can also describe the group $\bar{\omega}$ as follows. We define a metric ρ on Z^+ by the rule

$$\rho(x, y) = 1/n!\,, \qquad \text{where } n! \mid (x-y) \text{ and } (n+1)! \nmid (x-y),$$

with $\rho(x, x) = 0$. Then $\bar{\omega}$ is the completion of Z^+, with addition inherited from Z and extended by continuity. The proofs of these assertions can be found in [2]. In [1], H. Anzai and S. Kakutani called $\bar{\omega}$ the *universal monothetic Cantor Group*.

We denote the conjugate space of Q by Q^*; that is, Q^* is the Banach space of all continuous linear functionals defined on Q.

It is of interest to consider Q and Q^* in the context of the theory of almost periodic functions.

Let S be a locally compact abelian semigroup. We require S to have jointly continuous addition, but we do not require an identity. If $f \in C(S)$ and $a \in S$, define $T_a f$, the *translate* of f by a, by $T_a f(x) = f(x+a)$ for $x \in S$. We call $\{T_a f \mid a \in S\}$ the *orbit* of f. If the orbit of f is conditionally compact, we say that f is an *almost periodic* function on S, and we write $f \in AP(S)$.

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