THE DERIVED SETS OF A LINEAR SET

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For an arbitrary set E on the x-axis and an ordinal number b, let E^b denote the derived set of order b of E (the definition of the derived sets of a set, together with a discussion of many of their properties, was given by Cantor [2, pp. 98, 139-246]; for an excellent concise treatment of the material which is used here, see Baire [1, Chpt. III, especially §§ 32, 41, 42]). Since the set E^{Ω}, where Ω denotes the least nondenumerable ordinal number, is perfect, $E^b = E^{\Omega}$ for all $b > \Omega$. If a point x in E does not lie in E^{Ω}, the least ordinal b for which x does not lie in E^b is of the first kind; for if x lies in every set E^a $(a < b)$ and b is an ordinal of the second kind, then, since E^b is the intersection of the closed sets E^a $(a < b)$, the point x lies also in E^b. It follows that for every x in $E - E^{\Omega}$ there exists a greatest ordinal b for which x lies in E^b. This suggests the following definition of an ordinal-valued function $p(x, E)$. For an arbitrary fixed set E and each point x, let

$$p(x, E) = \Omega \text{ if } x \text{ lies in } E^{\Omega},$$

$$p(x, E) = 0 \text{ if } x \text{ does not lie in } E^1,$$

$$p(x, E) = b \text{ if } x \text{ lies in } E^b \text{ but not in } E^{b+1}.$$

It is the purpose of this note to characterize, among all the ordinal-valued functions $p(x)$, those functions for which there exists a point set E such that $p(x, E) \equiv p(x)$.

THEOREM. *If* $p(x)$ *is an ordinal-valued function defined on the* x-*axis, a necessary and sufficient condition for the existence of a point set* E *with* $p(x, E) = p(x)$ *is that* $p(x)$ *have the following properties:*

(1) *the set where* $p(x) = \Omega$ *is perfect, and the set where* $p(x) > \Omega$ *is empty;*

(2) *for each* b $(0 < b < \Omega)$, *the set where* $p(x) = b$ *is isolated;*

(3) *for each point* x_0, *the inequality* $p(x) \leq p(x_0)$ *holds throughout some neighborhood of* x_0;

(4) *if* $p(x_0) < \Omega$ *and* $a < p(x_0)$, *then* $p(x)$ *takes the value* a *in every neighborhood of* x_0.

The necessity of the condition is not new. Indeed, if there exists a set E such that $p(x, E) = p(x)$, then (1) follows from the fact that E^{Ω} is perfect, and (2) and (3) follow from the fact that every limit point of E^b belongs to E^{b+1} $(b \geq 1)$. Finally, if x_0 lies in an open interval I in which $p(x) < \Omega$, and if $p(x)$ does not take the value a in I, then $E^a = E^{a+1}$ in I; since $I \cap E^{\Omega}$ is empty, this implies that $I \cap E^a$ is also empty, and therefore $p(x) < a$ throughout I.

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