ON CLOSE-TO-CONVEX UNIVALENT FUNCTIONS

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1. INTRODUCTION

In a recent paper, W. Kaplan [3] introduced a class of univalent functions which he called close-to-convex. This class, which we shall call K, includes as proper subclasses the well-known convex functions, star maps, and Robertson's functions which are convex in a given direction [6]. In this note, we shall show that the normalized functions of class K have coefficients that verify the so-called Bieberbach conjecture, and we shall obtain a theorem of Study-type which is analogous to a theorem due to Carathéodory [2]. In addition, we introduce a class of functions we shall call close-to-star functions; these bear the same relation to Kaplan's close-to-convex functions that Robertson's analytic functions star-like in one direction bear to his analytic functions convex in some direction.

2. THE BIEBERBACH CONJECTURE FOR CLOSE-TO-CONVEX FUNCTIONS

If $f(z)$ is analytic for $|z| < 1$, and if $f'(z) \neq 0$ for $|z| < 1$, then $f(z)$ is said to be close-to-convex if and only if there exists a univalent convex function $\phi(z)$ such that

$$\Re \frac{f'(z)}{\phi'(z)} \geq 0 \tag{1}$$

holds in $|z| < 1$. The class of close-to-convex functions we shall denote by K. It was shown by Kaplan that (1) may be replaced by

$$\int_{\theta_1}^{\theta_2} \Re \left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta > -\pi, \tag{2}$$

which must hold for all $\theta_1 < \theta_2$ and for all $0 \leq r < 1$. For these functions we have the following result.

THEOREM 1. *Let* $f(z) \equiv \Sigma_0^\infty a_n z^n$ *be close-to-convex for* $|z| < 1$. *Then the coefficients satisfy the inequality*

$$|a_n| \leq n|a_1| \quad (n = 1, 2, 3, \cdots). \tag{3}$$

Proof. Since $f(z)$ is close-to-convex, there exists a univalent convex function $\phi(z)$ satisfying (1). Without loss of generality, we may assume that $f(z)$ and $\phi(z)$ are normalized, that is

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