APPROXIMATION BY ENTIRE FUNCTIONS

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1. INTRODUCTION

In 1927, Carleman [7] generalized the classical Weierstrass approximation theorem by proving that every function $Q(x)$ continuous for $-\infty < x < \infty$ can be uniformly approximated by an entire function. It is the purpose of the present paper to present several extensions of the Carleman theorem and to point out some of their applications. For the sake of completeness, a brief proof of the Carleman theorem is given in Section 2.

The first extension (Section 3) concerns the approximation of a function $Q(z)$, continuous on a subset C of a domain D, by a function $f(z)$ analytic in D; C is chosen as the union of a family of disjoint open arcs c_n $(n = 1, 2, \cdots)$ approaching the boundary of D individually and as a family.

In Section 4 it is shown that if $Q(x)$ has a continuous derivative for $-\infty < x < \infty$, then an entire function $f(z)$ exists such that both $f(x) - Q(x)$ and $f'(x) - Q'(x)$ tend to zero arbitrarily rapidly as $|x| \to \infty$; in Section 5 this is applied to approximation of paths by analytic paths.

The final section is devoted to the Dirichlet problem for the unit disc; existence of a "solution" is proved for a general class of nonintegrable boundary values and indeed for arbitrary measurable boundary values.

2. THE CARLEMAN APPROXIMATION THEOREM

THEOREM 1. *Let* $Q(x)$ *be a continuous complex-valued function of* x *for* $-\infty < x < \infty$. *Let* $E(x)$ *be continuous and positive for* $-\infty < x < \infty$. *Then there exists an entire function* $f(z)$ $(z = x + iy)$ *such that* $|f(x) - Q(x)| < E(x)$ *for* $-\infty < x < \infty$.

For this theorem, originally proved by Carleman [7], we give here a brief proof suggested by M. Brelot in a private communication.

Select constants E_n $(n = 0, 1, 2, \cdots)$ such that

$$E_0 = E_1 > E_2 > E_3 > \cdots > E_n > \cdots,$$

$$0 < E_n < E(x) \text{ for } n \leq |x| \leq n + 1.$$

Let $d_n = E_{n+1} - E_{n+2}$ $(n = -1, 0, 1, \cdots)$. A sequence of polynomials $f_n(z)$ is then chosen inductively as follows. First $f_0(z)$ is chosen (in accordance with the Weierstrass theorem) so that $|f_0(x) - Q(x)| < d_0$ for $|x| \leq 1$. When $f_n(z)$ has been chosen, then a function $g_{n+1}(z)$ is defined as $f_n(z)$ for $|z| \leq n + 1$ and as $Q(x)$ for z real and $n + 1 + 1/2 \leq |x| \leq n + 2$; $g_{n+1}(x)$ is also defined for $n + 1 \leq |x| \leq n + 1 + 1/2$ so as to remain continuous for $n + 1 \leq |x| \leq n + 2$ and so that $|g_{n+1}(x) - Q(x)| < d_n$ on these two intervals. By a theorem of Walsh ([14], p. 47), $f_{n+1}(z)$ can now be

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