A THEOREM OF FRIEDRICHS

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§1. Friedrichs [2] has given a characterization of the Lie elements among the set of noncommutative polynomials. A proof of the characterization theorem was also given by Magnus [3], who refers to other proofs by P. M. Cohn and D. Finkelstein. It is the purpose of the present paper to give a short proof of the theorem.

Let Φ be the free associative ring, over a field K of characteristic zero, of polynomials $F(x) = F(x_1, x_2, \cdots)$ in the noncommuting indeterminates $x_1, x_2, \cdots$. Let Λ be the K-submodule of Φ generated by the $x_1, x_2, \cdots$ under the operation of forming commutators $[G, H] = GH - HG$. A *Lie element* of Φ is a member of Λ.

THEOREM (Friedrichs). $F(x)$ *is a Lie polynomial if and only if the relations*

$$x_i' x_j'' = x_j'' x_i'$$

imply

(1) $$F(x' + x'') = F(x') + F(x'').$$

§2. Induction from Lie elements G, H to [G, H], together with linearity, establishes that (1) holds for every Lie element F.

For the converse, begin by introducing the left, right, and adjoint representations L, R, and $A = R - L$ of Φ. These are defined, on the free generators x_i, and for each element u of Φ, by the relations

$$uR(x_i) = ux_i,$$
$$uL(x_i) = x_i u,$$
$$uA(x_i) = ux_i - x_i u = [u, x_i].$$

Since the $R(x_i)$ commute with the $L(x_j)$, condition (1) on $F(x)$ implies that

$$uF(A(x)) = uF(R(x)) + uF(-L(x)).$$

Clearly $uF(R(x)) = uF(x)$, while $uF(-L(x)) = F(x)^*u$, where $F(x)^*$ is defined by the equation

$$(x_{i_1} x_{i_2} \cdots x_{i_n})^* = (-1)^n x_{i_n} \cdots x_{i_2} x_{i_1}$$

and the condition of linearity. Thus (1) gives

(2) $$uF(A(x)) = uF(x) + F(x)^* u.$$

Induction from G, H to [G, H] establishes that

(3) $$F^* = -F, \quad \textit{for each Lie element } F.$$

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