A GENERALIZATION OF INNER PRODUCT

R. K. Ritt

The purpose of this note is twofold: first, to show that if B is a real Banach space whose conjugate space B^* is uniformly convex [2], it is possible to define a real-valued function on $B \times B$ which for every $x \in B$ is a linear functional in y, for every $y \in B$ is a continuous (although not necessarily linear) function of x, and which is the usual inner product when B is a unitary (or Hilbert) space; second, to show that if B^* is strictly convex, a necessary and sufficient condition that B be unitary is that if $x \in B$, $y \in B$, and $\|x\| = \|y\| = 1$, then

$$\frac{d}{dt}\|x + ty\|\Big]_{t=0} \quad \text{and} \quad \frac{d}{dt}\|y + tx\|\Big]_{t=0}$$

exist and are equal. This condition is not new; James proved a similar result for Banach spaces of three or more dimensions [5, p. 283]. His proof depends on a generalization of orthogonality, ours on a generalization of inner product; even this generalization appears in a disguised form in James' paper; the elementary nature of our argument is the justification for its presentation here. We recall two definitions: A Banach space is *strictly convex* if $\|x + y\| = \|x\| + \|y\|$ implies that there exists a $\lambda \geq 0$ such that $x = \lambda y$; a Banach space is *uniformly convex* if

$$\|x_n\| = \|y_n\| = 1,$$

together with $\lim \|(x_n + y_n)/2\| = 1$, implies that $\lim \|x_n - y_n\| = 0$. This definition of uniform convexity is given by Hille [3, p. 11], and it can be shown to be equivalent to the definition originally given by Clarkson.

THEOREM. *Let* B *be a real Banach space whose conjugate space* B^* *is strictly convex. Then if* $x \in B$ $(x \neq 0)$, *there exists a unique* $\phi_x \in B^*$ *such that* $\phi_x(x) = 1$ *and* $\|\phi_x\| = 1/\|x\|$. *If* B^* *is uniformly convex, the mapping* $x \to \phi_x$ *is continuous.*

Proof. A slightly different version of the first statement has been proved by Pettis [7]. Briefly, if $x^*(x) = y^*(x) = 1$ and $\|x^*\| = \|y^*\| = 1/\|x\|$, then

$$\frac{1}{2}(x^* + y^*)(x) = 1;$$

thus $1/\|x\| \leq \|(x^* + y^*)/2\| \leq 1/\|x\|$, which contradicts the strict convexity of B^*, unless $x^* = y^*$.

To prove the second statement, we first observe that a uniformly convex space is reflexive [7] and that bounded sets are therefore sequentially compact [8]. Second, we prove that if $\{x_n^*\}$ is a sequence in B^* converging weakly to x_o^*, and if $\|x_n^*\|$ converges to $\|x_o^*\|$, then x_n^* converges uniformly to x_o^*. Since $x_n^*/\|x_n^*\|$ converges weakly to $x_o^*/\|x_o^*\|$, we can assume that $\|x_n^*\| = \|x_o^*\| = 1$. It remains to show that $\|(x_n^* + x_o^*)/2\|$ converges to 1.

Clearly $\|(x_n^* + x_o^*)/2\| \leq 1$, for all n. Let

$$x^{**} \in B^{**}, \quad \|x^{**}\| = 1, \quad x^{**}(x_o^*) = 1.$$

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