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Normal Embeddings of Semialgebraic Sets

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1. Introduction

This paper is devoted to some metric properties of semialgebraic sets with singularities. The metric theory of singularities considers sets as metric spaces. There are several classification problems in this theory; here we consider the problem of bi-Lipschitz classification.

Two metric spaces (X_1, d_1) and (X_2, d_2) are called *bi-Lipschitz equivalent* if there exist a homeomorphism $F: X_1 \to X_2$ and two positive constants K_1 and K_2 such that

$$K_1 d_1(x, y) \leq d_2(F(x), F(y)) \leq K_2 d_1(x, y)$$

for every $x, y \in X_1$. The homeomorphism F is called *a bi-Lipschitz map*. The bi-Lipschitz classification is stronger than topological and weaker than analytical classifications.

We can define two natural metrics on the same semialgebraic subset of $\mathbb{R}^n$: induced and length. The definition of the length metric came from differential geometry (see, for example, [G]). It is defined as the infimum of lengths of piecewise smooth curves connecting two given points. The Lipschitz classification in terms of the induced metric is more rigid: the equivalence in the induced metric implies the equivalence in the length metric, but not inversely. There exists a special type of sets—so-called normally embedded sets—such that these two classifications are equivalent. A set is *normally embedded* if the induced metric is equivalent to the length metric in the usual sense of metric spaces (see Definition 2.1). Every nonsingular compact semialgebraic subset is normally embedded, but the converse is not true.

The main result of the paper is the following normal embedding theorem: *Every compact semialgebraic set is bi-Lipschitz equivalent to some normally embedded semialgebraic set with respect to the length metric.* It is a metric analog of the normalization theorem [L] or of the desingularization theorem [H]. The proof is based on the so-called pancake decomposition (Section 2) created by Parusinski [P] and Kurdyka [K] (see also [KM] for details). Using a pancake decomposition we can define the pancake metric (Section 3), a semialgebraic metric equivalent

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