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Uniform Quotient Mappings of the Plane

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1. Introduction

Let X and Y be metric spaces. As is well known, a mapping $f: X \to Y$ is said to be *uniformly continuous* if there is a continuous increasing function $\Omega(r)$, $r \geq 0$ with $\Omega(0) = 0$, so that $d(f(u), f(v)) \leq \Omega(d(u, v))$ for all u and v; or, in other words, $f(\mathrm{B}_r(x)) \subset \mathrm{B}_{\Omega(r)}(f(x))$ for all $x \in X$ and $r > 0$. (We use $\mathrm{B}_r(x)$ to denote the open ball with radius r and center x in the appropriate space.) The mapping f is called *co-uniformly continuous* if there is a continuous increasing function $\omega(r)$, $r > 0$ with $\omega(r) > 0$ for $r > 0$, so that $\mathrm{B}_{\omega(r)}(f(x)) \subset f(\mathrm{B}_r(x))$. The continuity and monotonicity assumptions are made here for convenience and, if not assumed, can be achieved by changing the original functions $\Omega(r)$ and $\omega(r)$. The only necessary requirement is that the limit of $\Omega(r)$ is zero as $r \to 0$.

A surjective mapping f is said to be a *uniform quotient mapping* if it is uniformly continuous and co-uniformly continuous. In other words, f from X onto Y is a uniform quotient mapping if and only if $f \times f: X \times X \to Y \times Y$ maps the uniform neighborhoods of the diagonal in $X \times X$ onto the uniform neighborhoods of the diagonal in $Y \times Y$. Note that if $f: X \to Y$ is uniformly continuous and co-uniformly continuous then f (which of course is open) maps X to a closed set; hence the image of X is both closed and open. Consequently, if Y is connected then f is automatically surjective. Note also that if f is continuous and open and K is a compact subset of X, then for each $r > 0$ there is $\omega(r) > 0$ such that $\mathrm{B}_{\omega(r)}(f(x)) \subset f(\mathrm{B}_r(x))$ is satisfied for x in K. In particular, a continuous open mapping on a compact space is co-uniformly continuous. Finally, if f is uniformly continuous and co-uniformly continuous, then for all $Z \subset Y$ the restriction of f to $f^{-1}(Z)$, when considered as a mapping into Z, is also uniformly continuous and co-uniformly continuous; moreover, the image of every component of $f^{-1}(Z)$ is a component of Z provided that the balls of X are connected and $Z \subset f(X)$ is open. A discussion of the notion of co-uniform continuity and uniform quotient mappings (in the context of general uniform spaces) can be found in [J]. For normed spaces, the moduli always satisfy $\Omega(r) \geq Cr$ and $\omega(r) \leq cr$ for suitable C and c. If $\Omega(r) \leq Cr$ (more precisely, if Ω can be chosen to satisfy $\Omega(r) \leq Cr$ for some $0 < C < \infty$ and all $r > 0$), then we say that f is *Lipschitz*.

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