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Height Formulas for Homogeneous Varieties

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Dedicated to my teacher, William Fulton

In this paper we use classical Schubert calculus to evaluate the integral formula of Kaiser and Köhler [KK] for the Faltings height of certain homogeneous varieties in terms of combinatorial data, and we verify their conjecture for the size of the denominators.

1. Introduction

Consider a system of diophantine equations with integral coefficients which defines an arithmetic variety X in projective space $\mathbb{P}^n_{\mathbb{Z}}$. The Faltings height $h(X)$ of X is a measure of the arithmetic complexity of the system; it is an arithmetic analog of the geometric notion of the degree of a projective variety. The height $h(X)$ generalizes the classical height of a rational point of projective space, used by Siegel [S], Northcott [N] and Weil [W] to study questions of diophantine approximation. Faltings [F] defined $h(X)$ using the arithmetic intersection theory of Gillet and Soulé [GS2]; if $\overline{\mathcal{O}}(1)$ denotes the canonical hermitian line bundle on $\mathbb{P}^n$, then the height

$$h(X) = h_{\overline{\mathcal{O}}(1)}(X) = \widehat{\deg}(\hat{c}_1(\overline{\mathcal{O}}(1))^{\dim(X)} \mid X)$$

is the arithmetic degree of $X \subset \mathbb{P}^n$ with respect to $\overline{\mathcal{O}}(1)$. More generally, one has a notion of height of algebraic cycles with respect to hermitian line bundles; see [BGS, Sec. 3]. Our interest here is in explicit computations for these heights when $X = G/P$ is a homogeneous space of a Chevalley group G.

There are several alternative ways to identify the Faltings height $h(X)$. Although not as intrinsic as the above definition, they involve a more direct use of the equations in the system defining X. The approach by Philippon [Ph] uses an "alternative Mahler measure" of the Chow form of X. When X is a hypersurface defined by a homogeneous polynomial $f \in \mathbb{Z}[z_0, \dots, z_n]$, this gives

$$h(X) = \deg(f)h(\mathbb{P}^n) + \int_{S^{2n+1}} \log|f(z)|\,d\sigma, \tag{1}$$

where $d\sigma$ denotes the $U(n+1)$-invariant probability measure on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$; the Faltings height of projective space is given by

$$h(\mathbb{P}^n) = \frac{1}{2}\sum_{k=1}^{n} \mathcal{H}_k \tag{2}$$

(see also [BGS, Sec. 3.3.1]). Here $\mathcal{H}_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$ is a *harmonic number.*

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