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Intersection Multiplicities and Hilbert Polynomials

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Dedicated to W. Fulton

1. Introduction

In [11], Serre introduced a definition of intersection multiplicity for regular local rings, showed that it satisfied many of the properties which should hold for intersection multiplicities, and stated a number of conjectures. Of these conjectures, the only one that is still open is the positivity conjecture, which states that under certain conditions on dimension (we give a precise statement below), the intersection multiplicity will be positive. Recently, Gabber used a construction of de Jong to prove that these multiplicities are always nonnegative, thus establishing one of the conjectures. In his proof, Gabber constructed a scheme that can be represented by a bigraded ring and reduced the computation of intersection multiplicities to the computation of an Euler characteristic defined by modules over this ring. In this paper we define Hilbert polynomials for bigraded modules over this type of bigraded ring and show that the Euler characteristic can be computed using these Hilbert polynomials. We then use this construction to give a simple proof of a criterion for positivity proven in Kurano and Roberts [7]. Some of these ideas were discussed in Roberts [10]; however, the criterion we prove here was not included in that paper.

The outline of the paper is as follows. In Section 2 we recall the facts we need about the positivity conjecture and Gabber's construction. In Section 3 we prove the existence of Hilbert polynomials in the case we are considering; we then prove (Section 4) a reduction formula for dividing by a homogeneous element. In Section 5 we prove the basic relations between Hilbert polynomials and dimension. Finally, we prove the criterion for positivity in Section 6.

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2. Intersection Multiplicities and Gabber's Construction

Let R be a regular local ring of dimension d with maximal ideal $\mathfrak{m}$, and let $X = \operatorname{Spec}(R)$. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $\mathfrak{p} + \mathfrak{q}$ is $\mathfrak{m}$-primary or, equivalently, such that $R/\mathfrak{p} \otimes_R R/\mathfrak{q}$ is a module of finite length. Then the intersection multiplicity of $R/\mathfrak{p}$ and $R/\mathfrak{q}$ is defined to be

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