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Fundamental Groups of Rationally Connected Varieties

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Dedicated to William Fulton

1. Introduction

Let X be a smooth, projective, unirational variety, and let $U \subset X$ be an open set. The aim of this paper is to find a smooth rational curve $C \subset X$ such that the fundamental group of $C \cap U$ surjects onto the fundamental group of U. Following the methods of [K4] and [Co], a positive answer over $\mathbb{C}$ translates to a positive answer over any p-adic field. This gives a rather geometric proof of the theorem of [Hb] about the existence of Galois covers of the line over p-adic fields (1.4). We also obtain a slight generalization of the results of [Co] about the existence of certain torsors over open subsets of the line over p-adic fields (1.6).

If $U = X$ then $\pi_1(X)$ is trivial (cf. (2.3)), thus any rational curve C will do. If $X \setminus U$ is a divisor with normal crossings and if C intersects every irreducible component of $X \setminus U$ transversally, then the *normal* subgroup of $\pi_1(U)$ generated by the image of $\pi_1(C \cap U)$ equals $\pi_1(U)$ by a simple argument. (See e.g. the beginning of (4.2).) It is also not hard to produce rational curves C such that the image of $\pi_1(C \cap U)$ has finite index in $\pi_1(U)$ (cf. (3.3)). These results suggest that we are very close to a complete answer, but surjectivity is not obvious. Differences between surjectivity and finiteness of the index appear in many similar situation; see, for instance, [K1, Part I] or [NR].

The present proof relies on the machinery of rationally connected varieties developed in the papers [KMM1; KMM2; KMM3]. The relevant facts are recalled in Section 2.

The main geometric result is the following theorem.

Theorem 1.1. *Let K be an algebraically closed field of characteristic* 0, *and let X be a smooth projective variety over K that is rationally connected* (2.1). *Let $U \subset X$ be an open subset and $x_0 \in U$ a point. Then there is an open subset* $0 \in V \subset \mathbb{A}^1$ *and a morphism $f : V \to U$ such that $f(0) = x_0$ and*

$$\pi_1(V, 0) \twoheadrightarrow \pi_1(U, x_0) \text{ is surjective.}$$

Moreover, we can assume that the following also hold:

(1) $H^1(\mathbb{P}^1, \bar{f}^* T_X(-2)) = 0$, *where $\bar{f} : \mathbb{P}^1 \to X$ is the unique extension of f*;
(2) $\bar{f}$ *is an embedding if* $\dim X \geq 3$ *and an immersion if* $\dim X = 2$.

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