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A Construction of Irreducible GL(m) Representations

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Dedicated to Bill Fulton

1. Introduction

Let E be a finite-dimensional vector space over a field K. Fulton [3] has presented an elegant description of irreducible GL(E)-modules when K is of characteristic 0, a treatment that combines the classical approach in terms of products of determinants (see [4] for details and historical remarks) with a functorial approach. We briefly recall his construction.

Let $\{e_1, \dots, e_m\}$ be a basis for E, let $A = \{1, \dots, m\}$, and let λ be a partition. Consider a set $X = \{X_{i,a} \mid 1 \le i \le l(\lambda),\ a \in A\}$ of indeterminates over K. For a p-tuple $S = (a_1, \dots, a_p)$, $a_i \in A$, define $D_S = \det(X_{i,a_j})$, $1 \le i, j \le p$. The D_S are elements of the polynomial ring $K[X]$ in the $X_{i,a}$. An action of GL(m) on $K[X]$ is determined by $g \cdot X_{i,a} = \sum_{b \in A} g_{b,a} X_{i,b}$ for $g = (g_{b,c}) \in \mathrm{GL}(m)$.

For each S as just described we write $e_S = e_{a_1} \wedge \dots \wedge e_{a_p}$ for the corresponding element of the exterior power $\bigwedge^p E$. Let T be a filling of λ with entries from A. We associate with T an element $e_T \in \bigwedge^{\mu_1} E \otimes \dots \otimes \bigwedge^{\mu_h} E$, where μ is the conjugate of λ, by defining $e_T = e_{T_1} \otimes \dots \otimes e_{T_h}$ for $T_1, \dots, T_h$ columns of T.

We have a map of GL(m)-modules $\varphi_\lambda \colon \bigwedge^{\mu_1} E \otimes \dots \otimes \bigwedge^{\mu_h} E \to K[X]$ with $\varphi_\lambda(e_T) = D_T := D_{T_1} \cdots D_{T_h}$ for each filling T of λ.

The results we would like to quote from [3, Chap. 8] are as follows. If char $K = 0$, then:

(i) $E(\lambda) := \operatorname{Im} \varphi_\lambda \cong \bigwedge^{\mu_1} E \otimes \dots \otimes \bigwedge^{\mu_h} E / \operatorname{Ker} \varphi_\lambda$ is an irreducible GL(m)-module of highest weight λ if $l(\lambda) \le m$;
(ii) the set $\{D_T \mid T \text{ tableau}\}$ is a basis of $E(\lambda)$;
(iii) $\operatorname{Ker} \varphi_\lambda$ is generated by explicitly described elements that correspond to Sylvester's identities among the D_T.

In this paper we present a similar approach with exterior powers replaced by symmetric powers. It requires considering exterior algebra indeterminates instead of polynomial indeterminates and leads to a new construction of irreducible GL(m)-modules. A combination of both approaches can be used to construct in the same vein tensor representations of general linear Lie superalgebras (see [6]).

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