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Mori Dream Spaces and GIT

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Dedicated to Bill Fulton

An important advance in algebraic geometry in the last ten years is the theory of variation of geometric invariant theory (VGIT) quotient; see [BP; DH; H1; T]. Several authors have observed that VGIT has implications for birational geometry—for example, it gives natural examples of Mori flips and contractions [DH; R2; T]. In this paper we observe that the connection is quite fundamental: Mori theory is, at an almost tautological level, an instance of VGIT; see (2.14). Here are more details.

Given a projective variety X, a natural problem is to understand the collection of all morphisms (with connected fibres) from X to other projective varieties. Ideally one would like to decompose each map into simple steps and parameterize the possibilities, both for the maps and for the factorizations of each map. An important insight, principally of Reid and Mori, is that the picture is often simplified if one allows, in addition to morphisms, *small modifications*—that is, rational maps that are isomorphisms in codimension 1. With this extension, a natural framework is the category of rational contractions. In many cases there is a nice combinatorial parameterization given by a decomposition of a convex polyhedral cone, the cone of effective divisors $\overline{NE}^1(X)$, into convex polyhedral chambers, which we call Mori chambers. Instances of this structure have been studied in various circumstances: The existence of such a parameterizing decomposition for Calabi–Yau manifolds was conjectured by Morrison [M], motivated by ideas in mirror symmetry. The conjecture was proven in dimension 3 by Kawamata [Kaw]. Oda and Park [OP] study the decomposition for toric varieties, motivated by questions in combinatorics. Shokurov studied such a decomposition for parameterizing log-minimal models. In geometric invariant theory there is a similar combinatorial structure, a decomposition of the G-ample cone into GIT chambers parameterizing GIT quotients; see [DH]. The main observation of this paper is that whenever a good Mori chamber decomposition exists, it is in a natural way a GIT decomposition.

The main goal of this paper is to study varieties X with a good Mori chamber decomposition (see Section 1 for the meaning of “good”). We call such varieties *Mori dream spaces.* There turn out to be many examples, including quasi-smooth projective toric (or, more generally, spherical) varieties, many GIT quotients, and log Fano 3-folds. We will show that a Mori dream space is, in a natural way, a GIT quotient of affine variety by a torus in a manner generalizing Cox’s construction

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