**Michigan Math. J. 48 (2000)**

# Localization and Test Exponents for Tight Closure

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*Dedicated to William Fulton*

## 1. Introduction

We introduce the notion of a *test exponent* for tight closure and explore its relationship with the problem of showing that tight closure commutes with localization, a long-standing open question. Roughly speaking, test exponents exist if and only if tight closure commutes with localization; mild conditions on the ring are needed to prove this. We give other, independent, conditions that are necessary and sufficient for tight closure to commute with localization in the general case, in terms of behavior of certain associated primes and behavior of exponents needed to annihilate local cohomology. Although certain related conditions (the ones given here are weaker) were previously known to be sufficient, these are the first conditions of this type that are actually equivalent.

The difficult calculation of Section 4 uses associativity of multiplicities and many other tools to show that sufficient conditions for localization to commute with tight closure can be given in which asymptotic statements about lengths of modules defined using the iterates of the Frobenius endomorphism replace the finiteness conditions on sets of primes introduced in Section 3. The result is local and requires special conditions on the rings: one is that countable prime avoidance holds. This is not a very restrictive condition, however; it suffices, for example, for the ring to contain an uncountable field. Countable prime avoidance also holds in any complete local ring. But we also need the existence of a *strong* test ideal (see the beginning of Section 4). We expect that, in the long run, this condition will also turn out not to be very restrictive: a strong test ideal for a reduced ring is known to exist if every irreducible component of Spec $R$ has a resolution of singularities obtained by blowing up an ideal that defines the singular locus, and it is expected that this is always true in the excellent case. Moreover, by very recent results, strong test ideals always exist for complete reduced local rings.

We note that the reader may find other results related to localization of tight closure in [AHH; Hu3; K1; K2; LS; V1; V2].

Received March 24, 2000. Revision received April 19, 2000.
The authors were supported in part by grants from the National Science Foundation.
The second author wishes to thank the Max Planck Institute in Bonn and the Fulbright Foundation for partial support of the research for this manuscript.