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Global Structures on CR Manifolds via Nash Blow-Ups

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Dedicated to William Fulton on his sixtieth birthday

1. Introduction

Let X be a compact real $(2n+2)$-dimensional submanifold of the complex space C^{n+2}. For generic such X at all but a finite number of points, the tangent space of X will have a $2n$-dimensional subspace H that inherits a complex structure from the ambient C^{n+2}. There are, though, topological obstructions preventing the subspaces H from forming a subbundle of the tangent bundle TX. The existence of such obstructions was shown by Wells [35]. Lai [26] gave an explicit description of these obstructions.

There has recently been a lot of work on determining when two CR structures are locally equivalent, subject to various restrictions on dimension and conditions on the Levi form. There is the work of Beloshapka [1; 2], Ebenfelt [10; 11], Ezhov and Isaev [12], Ezhov, Isaev, and Schmalz [13], Ezhov and Schmalz [14; 15; 16; 17], Garrity and Mizner [18; 19], Le [27], Mizner [28], and Schmalz and Slovak [29]. These works concentrate on the understanding of the Levi form, a vector-valued Hermitian form at each point mapping $H \times H$ to TX/H.

All of these techniques and methods for producing local invariants break down for compact manifolds. What has prevented people from applying standard tools from differential geometry to understand the obstructions preventing the extensions of these local invariants to global invariants has been that the subbundle H is not a true subbundle. All of the local calculations depend on H, the part of the tangent bundle inheriting a complex structure from C^{n+2}, having real dimension $2n$. For a compact X, there will be points (the *complex jump points,* which we will denote by $\mathcal{J}$) where the H will have real dimension $2n+2$. The existence of these points is what prevents any easy attempt to extend local invariants to global ones.

We use a version of the Nash blow-up to replace X, subject to certain natural conditions, with a smooth manifold $\tilde{X}$ so that there is a natural map $\pi : \tilde{X} \to X$ with π an isomorphism from $\tilde{X} - \pi^{-1}\mathcal{J}$ to $X - \mathcal{J}$ and so that there is a complex rank-n vector bundle $\tilde{H}$ on $\tilde{X}$ such that $\tilde{H}$ pushes forward to the bundle H on $X - \mathcal{J}$. Thus global calculations can now be performed.

The method presented here is to show that there is a natural map (a version of the Gauss map) from $X - \mathcal{J}$ to a flag manifold F. The Nash blow-up is the closure

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