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Poincaré Duality and Equivariant (Co)homology

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To Bill Fulton for his 60th birthday

Introduction

Let X be a compact complex algebraic variety of pure dimension n whose Betti numbers vanish in all odd degrees. Then the cohomology ring $H^*(X)$ with complex coefficients is a commutative, positively graded algebra, of finite dimension as a complex vector space. It is well known that the dualizing module (in the sense of commutative algebra, see e.g. [8]) of $H^*(X)$ is the homology $H_*(X)$; moreover, $H^*(X)$ is Gorenstein if and only if X satisfies Poincaré duality. This holds if X is smooth or, more generally, rationally smooth; that is, the local cohomology at any point is the same as the local cohomology of complex affine n-space (see [17] for other characterizations).

We shall generalize these observations to the richer setting of equivariant homology and cohomology, with applications to Coxeter groups. Assume that a d-dimensional torus T acts on X with isolated fixed points (examples include rationally smooth projective varieties where a complex reductive group acts with finitely many orbits, Schubert varieties, and varieties of complete flags fixed by a given linear transformation). Then the equivariant cohomology ring $H_T^*(X)$ with complex coefficients is positively graded, commutative and reduced; it is a free module of finite rank over the equivariant cohomology ring of the point. The latter is a polynomial ring in d variables. Thus, the ring $H_T^*(X)$ is Cohen–Macaulay. We show that several topological invariants of the T-variety X can be read off that ring.

Specifically, restriction to the T-fixed point set $H_T^*(X) \to H_T^*(X^T)$ is the normalization of $H_T^*(X)$. It follows that the complex affine algebraic variety $V(X)$ associated to $H_T^*(X)$ is a finite union of copies of the Lie algebra of T, glued along rational hyperplanes (Proposition 2). The dualizing module of $H_T^*(X)$ turns out to be the equivariant Borel–Moore homology $H_*^T(X)$ (Proposition 1); it admits a more concrete description in terms of regular differential forms on $V(X)$ (Proposition 3). On the other hand, the conductor of $H_T^*(X)$ in its normalization $H_T^*(X^T)$ is closely related to equivariant cohomology with support in X^T, and also to equivariant multiplicities; the latter are uniquely determined by the abstract ring $H_T^*(X)$, up to a common scalar multiple (Section 3).

These considerations yield the following linear inequalities for the Betti numbers of a variety X as above, if all equivariant multiplicities are nonzero:

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