Michigan Math. J. 48 (2000)

Determinantal Hypersurfaces

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Introduction

(0.1) In this paper we discuss which homogeneous form on $\mathbf{P}^n$ can be written as the determinant of a matrix with homogeneous entries (possibly symmetric), or the pfaffian of a skew-symmetric matrix. This question has been considered in various particular cases (see the historical comments that follow), and we believe that the general result is well-known to the experts; but we have been unable to find it in the literature. The aim of this paper is to fill this gap.

We will discuss at the outset the general structure theorems; roughly, they show that expressing a homogeneous form F as a determinant (resp. a pfaffian) is equivalent to produce a line bundle (resp. a rank-2 vector bundle) of a certain type on the hypersurface $\mathrm{F} = 0$. The rest of the paper consists of applications. We have restricted our attention to *smooth* hypersurfaces; in fact, we are particularly interested in the case when the *generic* form of degree d in $\mathbf{P}^n$ can be written in one of the above forms. When this is the case, the moduli space of pairs (X, E), where X is a smooth hypersurface of degree d in $\mathbf{P}^n$ and E a rank-1 or rank-2 vector bundle satisfying appropriate conditions, appears as a quotient of an open subset of a certain vector space of matrices; in particular, this moduli space is *unirational*. This is true, for instance, of the universal family of Jacobians of plane curves (Corollary 3.6), and of intermediate Jacobians of cubic threefolds (Corollary 8.8).

Unfortunately, this situation does not occur very frequently: we will show that only curves and cubic surfaces generically admit a determinantal equation. The situation is slightly better for pfaffians: plane curves of any degree, surfaces of degree ≤ 15, and threefolds of degree ≤ 5 can be generically defined by a linear pfaffian.

(0.2) Historical Comments. The representation of curves and surfaces of small degree as linear determinants is a classical subject. The case of cubic surfaces was already known by the middle of the last century [G]; other examples of curves and surfaces are treated in [S]. The general homogeneous forms that can be expressed as linear determinants are determined in [D]. A modern treatment for plane curves appears in [CT]; the result has been rediscovered a number of times since then.

Received January 5, 2000. Revision received April 28, 2000.