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The Bergman Metric in the Normal Direction: A Counterexample

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1. Introduction

Let $D \subset\subset \mathbb{C}^n$ be a bounded domain. By $B_D(z; X)$ we denote its Bergman metric and by $d_B(z, w)$ the distance function associated to B_D. The question of the completeness of D with respect to d_B has found much interest. Kobayashi [13; 14] proved criteria for the Bergman completeness of a bounded domain D that are based on a representation of d_B by means of the Fubini–Study metric in the projective space $\mathbb{P}(H^2(D))$ over the Hilbert space $H^2(D)$ of all holomorphic functions on D that are square integrable with respect to Lebesgue measure. Almost all qualitative completeness results for the Bergman metric were obtained by means of this criterion (see e.g. [1; 11; 12; 16]).

Another interesting (yet more refined) method for studying the Bergman completeness of D consists of looking for quantitative estimates for d_B and B_D implying it. In this direction, a very general result was obtained by Diederich and Ohsawa [8], who proved that—for those hyperconvex domains admitting a plurisubharmonic exhaustion function ρ satisfying

$$c \operatorname{dist}(\cdot, \partial D)^m \le |\rho| \le C \operatorname{dist}(\cdot, \partial D)^{1/m}$$

with suitable constants $c, C, m > 0$—the Bergman distance grows at least like a constant times $\log\log(1/\operatorname{dist}(z, \partial D))$ for z sufficiently close to ∂D. This result applies in particular to all finite intersections of C^2-smooth pseudoconvex domains.

Let now $D = \{r < 0\}$ be a bounded pseudoconvex domain with smooth boundary and let $z^0 \in \partial D$. In this paper we study the following related question on the boundary behavior of the Bergman metric B_D near z^0:

Does there exist a constant $C > 0$ and an open neighborhood $U \ni z^0$ such that, for all directions $X \in \mathbb{C}^n$, one has the lower bound

$$B_D(z; X) \ge C \frac{|(\partial r(z), X)|}{|r(z)|} \tag{1}$$

on $D \cap U$?

Here $(\partial r(z), X) = \sum_{j=1}^n X_j \partial r/\partial z_j(z)$.

The inequality (1) has long been known to be true under certain additional hypotheses on the domain D. It holds for example when z^0 is strongly pseudoconvex

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