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Singular Integrals Related to Homogeneous Mappings

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1. Introduction

Let $n, m \in \mathbf{N}$ and $d = (d_1, \dots, d_m) \in \mathbf{R}^m$. Define the family of dilations $\{\delta_t\}_{t>0}$ on $\mathbf{R}^m$ by

$$\delta_t(x_1, \dots, x_m) = (t^{d_1}x_1, \dots, t^{d_m}x_m). \tag{1}$$

We say that $\Phi\colon \mathbf{R}^n \to \mathbf{R}^m$ is a (nonisotropic) homogeneous mapping of degree d if

$$\Phi(ty) = \delta_t(\Phi(y)) \tag{2}$$

holds for all $t > 0$ and $y \in \mathbf{R}^n$.

Let $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbf{R}^n$ that is equipped with the normalized Lebesgue measure $d\sigma$. For a Calderón–Zygmund kernel on $\mathbf{R}^n$,

$$K(y) = \frac{\Omega(y)}{|y|^n}, \tag{3}$$

where Ω is homogeneous of degree 0 and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(y)\, d\sigma(y) = 0; \tag{4}$$

we define the singular integral operator $T_{\Omega,\Phi}$ on $\mathbf{R}^m$ by

$$(T_{\Omega,\Phi} f)(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x - \Phi(y))K(y)\, dy \tag{5}$$

for $x \in \mathbf{R}^m$.

The operators defined in (5) have their roots in the classical Calderón–Zygmund operators

$$(T_{\Omega,I} f)(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x - y)K(y)\, dy, \tag{6}$$

which corresponds to $n = m$, $d = (1, \dots, 1)$, and $\Phi = I = \mathrm{id}_{R^n \to R^n}$. In their fundamental work on the theory of singular integrals, Calderón and Zygmund [1] proved that the operators $T_{\Omega,I}$ in (6) are bounded on L^p for $1 < p < \infty$ if $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$. Their result is nearly optimal in the sense that the space $L\log^+ L(\mathbf{S}^{n-1})$ cannot be replaced by any other Orlicz space $L^\phi(\mathbf{S}^{n-1})$ with a ϕ that is increasing and satisfies

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