# Global Plurisubharmonic Defining Functions

## Rachid Belhachemi & Alan Noell

## 1. Introduction

Given a bounded pseudoconvex domain with smooth (i.e., infinitely differentiable) boundary in $\mathbf{C}^n$, the Levi condition ensures that, for every defining function, the Levi form restricted to complex tangent vectors is positive semi-definite at each boundary point. Here we study the stronger condition that there exists a defining function plurisubharmonic on the boundary in the sense that, at each boundary point, the Levi form is positive semi-definite on *all* complex vectors. For strongly pseudoconvex domains (i.e., when the Levi form is positive definite on complex tangent vectors) it is elementary that there is a strongly plurisubharmonic defining function. For weakly pseudoconvex domains, a defining function plurisubharmonic on the boundary need not exist. The well-known "worm" domain introduced by Diederich and Fornæss in [7] has no such defining function, nor does the version of that domain with real-analytic boundary defined by Fornæss in [8]. In fact, a defining function plurisubharmonic on the boundary need not exist even locally. Fornæss gave such an example with only smooth boundary in [8], and later Behrens [1] gave a simpler example: There is a domain with a polynomial defining function such that (a) the Levi form degenerates at only one boundary point but (b) near this point, there is no local defining function plurisubharmonic on the boundary. Her example shows that any sufficient condition for the existence of a local defining function plurisubharmonic on the boundary must involve more than merely the structure of the degeneracy set of the Levi form.

Here we are interested in conditions under which the existence at each boundary point of a local defining function plurisubharmonic on the boundary implies the existence of a global defining function with this property. In this setting the structure of the degeneracy set enters naturally, since an obstruction can exist in the attempt to patch local defining functions along this set. This situation is illustrated by the real-analytic version of the "worm" domain constructed by Fornæss in [8], which does have local defining functions of the desired type even though (as just mentioned) there is no such global function. In this example the degeneracy set is a curve whose tangent space at each point is contained in the null space of the Levi form restricted to complex tangent vectors. Our condition that the domain be linearly regular (as defined in Section 2) rules out the existence of such curves, and this condition, along with the existence of local defining functions