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Local Hulls of Unions of Totally Real Graphs Lying in Real Analytic Hypersurfaces

Nguyen Quang Dieu

I. Introduction

Let Y be a compact subset of $\mathbf{C}^n$ and let $\hat{Y}$ denote the polynomial convex hull of Y, that is,

$$\hat{Y} = \left\{ z \in \mathbf{C}^n : |Q(z)| \le \max_Y |Q| \text{ for every polynomial } Q \text{ on } \mathbf{C}^n \right\}.$$

We say that Y is polynomially convex if $\hat{Y} = Y$. A closed subset F of $\mathbf{C}^2$ is called *locally polynomially convex* (LPC for short) at $a \in F$ if there exists $r > 0$ such that the intersection $\bar{\mathbf{B}}(a, r) \cap F$ is polynomially convex. Since the intersection of two polynomially convex sets is again polynomially convex, a polynomially convex set is LPC everywhere, but the converse is known to be false. Indeed, by classic work of Wermer, if X is a $\mathcal{C}^1$ totally real submanifold of $\mathbf{C}^n$ (i.e., if the real tangent space $T_a X$ contains no complex line for every $a \in X$) then X is LPC everywhere. But Wermer constructed a totally real embedded disk that bounds an analytic disk and hence is not (globally) polynomially convex (see e.g. [FS]).

To begin to consider nonsmooth varieties, it seems natural to study the local polynomial convexity of the union of two totally real submanifolds in $\mathbf{C}^2$. In the case where the manifolds are totally real planes in $\mathbf{C}^n$, a complete picture was obtained by Weinstock (see [W]).

Here we examine the case where the manifolds in question are graphs whose tangent planes at the origin meet only at one point. More precisely, let M_1 and M_2 be two totally real graphs in $\mathbf{C}^2$ such that $T_0 M_1 \cap T_0 M_2 = \{0\}$. We ask under what conditions the union is LPC at 0, and if it is not LPC then we will try to describe the hull near the origin. It should be observed that if $T_0 M_1 \cup T_0 M_2$ is LPC at 0 then, by an implicit result of Forstneric and Stout [FS], the union $M_1 \cup M_2$ is LPC at 0. After a linear change of coordinates, the case where $T_0 M_1 \cup T_0 M_2$ is *not* LPC at 0 reduces to

$$M_1 = \{(z, \bar{z} + \varphi_1(z))\} \quad \text{and} \quad M_2 = \{(z, \lambda\bar{z} + \varphi_2(z))\}, \tag{$*$}$$

where $\lambda > 0$, $\lambda \ne 1$, and φ_i $(i = 1, 2)$ are functions of class $\mathcal{C}^1$ in a neighborhood of 0 that satisfy $\varphi_i(0) = \partial\varphi_i/\partial z(0) = \partial\varphi_i/\partial\bar{z}(0) = 0$.

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