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Nonlinear Hyperbolic Smoothing at a Focal Point

Jean-Luc Joly, Guy Metivier, & Jeffrey Rauch

1. Introduction

We construct real-valued finite energy solutions of the dissipative nonlinear wave equation

$$\Box u + |u_t|^{h-1}u_t = 0, \quad \Box := \partial_t^2 - \Delta_x, \ 1 < h \in \mathbb{R}, \tag{1.1}$$

which have singularities that are partially smoothed after a focus. Here $\{t, x\} \in \mathbb{R}^{1+d}$, with spatial dimension $d \geq 2$.

A striking classical result of Lions and Strauss [LS] shows that (1.1) is a well-behaved evolution equation in $t \geq 0$ in all dimensions. Two underlying estimates are used in establishing this result. The first is that solutions have nonincreasing energy. With

$$E(u, t) := \int_{\mathbb{R}^d} \frac{u_t^2}{2} + \frac{|\nabla_x u|^2}{2}\, dx, \tag{1.2}$$

one has

$$E(u, t) = E(u, 0) - \int_0^T \int_{\mathbb{R}^d} \frac{|u_t|^{h+1}}{h+1}\, dx\, dt \leq E(u, 0). \tag{1.3}$$

More generally, one has a contractivity estimate that relies on the monotonicity of the nonlinear function

$$F_h(s) := |s^{h-1}|s.$$

Precisely,

$$\begin{aligned} E(u - v, t) &= E(u - v, 0) - \int_0^T \int_{\mathbb{R}^d} (u_t - v_t)(F_h(u_t) - F_h(v_t))\, dx\, dt \\ &\leq E(u - v, 0). \end{aligned} \tag{1.4}$$

The energy dissipation identity is the case $v = 0$ of the contractivity identity. These estimates lead to the following fundamental results of Lions and Strauss.

Theorem 1.1 [LS]. *If* $\{f, g\} \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ *then there is a unique solution* u *to* (1.1) *with*

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