Light-Cone Expansion of the Dirac Sea to First Order in the External Potential

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1. Introduction

In relativistic quantum mechanics, the problem of the unphysical negative-energy solutions of the Dirac equation is solved by the conception that all negative-energy states are occupied in the vacuum forming the so-called Dirac sea. In [1], the Dirac sea was constructed for the Dirac equation with general interaction in terms of a formal power series in the external potential. In the present paper, we turn our attention to a single Feynman diagram of this perturbation expansion. More precisely, we will analyze the contribution to first order in the potential and derive explicit formulas for the Dirac sea in position space. Since this analysis does not require a detailed knowledge of the perturbation expansion for the Dirac sea, we can make this paper self-consistent by giving a brief introduction to the mathematical problem.

In the vacuum, the Dirac sea is characterized by the integral over the lower mass shell

$$P(x, y) = \int \frac{d^4p}{(2\pi)^4} (\not{p} + m)\, \delta(p^2 - m^2)\, \Theta(-p^0)\, e^{-ip(x-y)} \tag{1.1}$$

(Θ is the Heavyside function, $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ otherwise); $P(x, y)$ is a tempered distribution that solves the free Dirac equation $(i\not{\partial}_x - m)P(x, y) = 0$. In the case with interaction, the Dirac sea is accordingly described by a tempered distribution $\tilde{P}(x, y)$ being a solution of the Dirac equation

$$(i\not{\partial}_x + \mathcal{B}(x) - m)\tilde{P}(x, y) = 0, \tag{1.2}$$

where $\mathcal{B}$ is composed of the classical bosonic potentials. We assume $\mathcal{B}$ to be a 4×4 matrix potential satisfying the condition $\gamma^0 \mathcal{B}(x)^\dagger \gamma^0 = \mathcal{B}(x)$ ("$\dagger$" denotes the transposed, complex conjugated matrix). We can thus decompose it in the form

$$\mathcal{B} = e\not{A} + e\gamma^5 \not{B} + \Phi + i\gamma^5 \Xi + \sigma_{jk} H^{jk} \tag{1.3}$$

with the electromagnetic potential A_j, an axial potential B_j, scalar and pseudoscalar potentials Φ and Ξ, and a bilinear potential H^{jk} (see e.g. [7] for a discussion of these potentials). In Appendix B, it is shown how the results can be extended to an external gravitational field.

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