On the Argument Oscillation of Conformal Maps

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathbf{T}$ its boundary. We shall consider (injective) conformal maps f of $\mathbb{D}$ into $\mathbb{C}$. For $\zeta \in \mathbf{T}$ we denote by $f(\zeta)$ the angular (= radial) limit if it exists and is finite. This holds for almost all $\zeta \in \mathbf{T}$; even the exceptional set has zero logarithmic capacity, by the well-known Beurling theorem (see [Be; Po2, p. 215]). Furthermore, the set $\{\zeta \in \mathbf{T} : f(\zeta) = a\}$ has zero capacity for every $a \in \mathbb{C}$ [Du; Po2, p. 219]. A stronger condition is that f is continuous at ζ; that is

$$f(z) \to f(\zeta) \quad \text{as } z \to \zeta,\ z \in \mathbb{D}. \tag{1.1}$$

Suppose now that the angular limit $f(\zeta) \neq \infty$ exists at $\zeta \in \mathbf{T}$. The function

$$g_\zeta(z) = \log[f(z) - f(\zeta)], \quad z \in \mathbb{D}, \tag{1.2}$$

is analytic and univalent in $\mathbb{D}$ for any branch of the logarithm. It therefore has finite angular limits at all points except a set of zero capacity. For convenience, throughout this paper we will write

$$E(\zeta) = \{\zeta' \in \mathbf{T} : f(\zeta') \text{ and } g_\zeta(\zeta') \text{ exist and are finite}\}.$$

Thus $\mathbf{T} \setminus E(\zeta)$ has zero capacity. Then we define the argument by

$$\arg[f(z) - f(\zeta)] = \begin{cases} \operatorname{Im} g_\zeta(z) & \text{for } z \in \mathbb{D}, \\ \lim\limits_{r \to 1} \operatorname{Im} g_\zeta(re^{it}) & \text{for } z = e^{it} \in E(\zeta). \end{cases} \tag{1.3}$$

We also consider the analytic function

$$h(z) = \log \frac{f(z) - f(\zeta)}{z - \zeta} = g_\zeta(z) - \log(z - \zeta), \quad z \in \mathbb{D} \cup E(\zeta). \tag{1.4}$$

In the unit disk, we always use the branch of the logarithm determined by

$$\theta + \pi/2 < \arg(z - \zeta) < \theta + 3\pi/2, \quad z \in \mathbb{D} \cup E(\zeta),\ \zeta = e^{i\theta}.$$

The function h is a Bloch function [Po2, p. 173]. We will make frequent use of the harmonic function

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