Pseudodifferential Operators with Homogeneous Symbols

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1. Introduction

The study of pseudodifferential operators with symbols in the exotic classes $S^m_{1,1}$ has received a lot of attention. These are operators of the form

$$(Tf)(x) = \int_{\mathbf{R^n}} e^{ix\cdot\xi} a(x,\xi)\hat{f}(\xi)\,d\xi,$$

where the symbol $a(x,\xi)$ is a $C^\infty(\mathbf{R^n} \times \mathbf{R^n})$ function satisfying

$$|\partial_\xi^\beta \partial_x^\gamma a(x,\xi)| \le C_{\beta,\gamma}(1+|\xi|)^{m-|\beta|+|\gamma|},$$

for all β and γ n-tuples of nonnegative integers. The interest in such operators is due in part to the role they play in the paradifferential calculus of Bony [1]. The fact that not all such operators of order zero are bounded on L^2 complicates their study. Nevertheless, the exotic pseudodifferential operators do preserve spaces of smooth functions. See, for example, Meyer [12], Paivarinta [14], Bourdaud [2], as well as Stein [16] and the references therein.

The continuity results are often obtained by making use of the so-called singular integral realization of the operators. This involves proving estimates on the Schwartz kernels of the pseudodifferential operators similar to those of the kernels of Calderón–Zygmund operators. There is, however, an alternative approach working directly with the symbols of the pseudodifferential operators. This approach has been pursued by Hörmander in [9] and [10] for L^2-based Sobolev spaces. The ideas in those papers combined with wavelets techniques were later extended by Torres [17] to L^p-based Sobolev spaces and other more general spaces of smooth functions.

In this note we consider C^∞ symbols $a(x,\xi)$ in $\mathbf{R^n} \times (\mathbf{R^n} \setminus \{\mathbf{0}\})$ that satisfy the following conditions: For all n-tuples of nonnegative integers β and γ there exist positive constants $C_{\beta,\gamma}$ such that

$$|\partial_\xi^\beta \partial_x^\gamma a(x,\xi)| \le C_{\beta,\gamma}|\xi|^{m-|\beta|+|\gamma|} \tag{1}$$

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