On Linear and Residual Properties of Graph Products

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1. Introduction

Graph groups are groups with presentations where the only relators are commutators of the generators. Graph groups were first investigated by Baudisch [1], and much subsequent foundational work was done by Droms, B. Servatius, and H. Servatius [3; 4; 5]. Later, the more general construction of *graph products* (Definition 2.1) was introduced and developed by Green [7]. (Graph products are to free products as graph groups are to free groups.) Graph groups have also been of recent interest because of their geometric properties (Hermiller and Meier [8] and VanWyk [13]) and the cohomological properties of their subgroups (Bestvina and Brady [2]).

In this paper, by embedding graph products in Coxeter groups, we obtain short proofs of several fundamental properties of graph products. Specifically, after listing some preliminary definitions and results in Section 2, we show in Section 3 that the graph product of subgroups of Coxeter groups is a subgroup of a Coxeter group (Theorem 3.2). It follows that many classes of graph products are linear, including graph groups (a result of Humphries [11]) and that the graph product of residually finite groups is residually finite (a result of Green [7]). In Section 4, we also include a new and more geometric proof of Green's normal form theorem for graph products. Finally, in Section 5, we list some related open problems.

2. Graph Products

In this section, we review some basic definitions and results on graph products.

For a simplicial graph Γ, we let Γ^0 denote the vertices of Γ, we let Γ^1 denote the edges of Γ, and we let $[v, w]$ denote the edge between the vertices v and w.

Definition 2.1. Let Γ be a finite simplicial graph, and for each $v \in \Gamma^0$ let G_v be a group called the *vertex group* of v. The graph product ΓG_v is defined to be the free product of the G_v, subject to the relations

$$[g_v, g_w] = 1 \quad \text{for all } g_v \in G_v,\ g_w \in G_w \text{ such that } [v, w] \in \Gamma^1. \tag{1}$$

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