# Componentwise Linear Ideals and Golod Rings

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*Dedicated to Jack Eagon on the occasion of his 65th birthday*

## 1. Introduction

Let $A = K[x_1, \dots, x_n]$ be a polynomial ring over a field $K$, and let $R = A/I$ be the quotient of $A$ by an ideal $I \subset A$ that is homogeneous with respect to the standard grading in which $\deg(x_i) = 1$. When $I$ is generated by square-free monomials, it is traditional to associate with it a certain simplicial complex $\Delta$, for which $I = I_\Delta$ is the *Stanley–Reisner ideal* of $\Delta$ and $R = K[\Delta] = A/I_\Delta$ is the *Stanley–Reisner ring* or *face ring.* The definition of $\Delta$ as a simplicial complex on vertex set $[n] := \{1, 2, \dots, n\}$ is straightforward: the minimal non-faces of $\Delta$ are defined to be the supports of the minimal square-free monomial generators of $I$.

Many of the ring-theoretic properties of $I_\Delta$ then translate into combinatorial and topological properties of $\Delta$ (see [14, Chap. II]). In particular, a celebrated formula of Hochster [14, Thm. II.4.8] describes $\mathrm{Tor}^A_{\cdot}(R, K)$ in terms of the homology of the full subcomplexes of $\Delta$. Here $K$ is considered the trivial $A$-module $K = A/\mathfrak{m}$ for $\mathfrak{m} = (x_1, \dots, x_n)$. It is well known that the dimensions of these $K$-vector spaces $\mathrm{Tor}^A_{\cdot}(R, K)$ give the ranks of the resolvents in the finite minimal free resolution of $R$ as an $A$-module.

In a series of recent papers, beginning with [8] and subsequently [9; 15; 13], it has been recognized that, for square-free monomial ideals $I = I_\Delta$, there is another simplicial complex $\Delta^*$ which can be even more convenient for understanding free $A$-resolutions of $R$. The complex $\Delta^*$, which from now on we will call the *Eagon complex* of $I = I_\Delta$, carries equivalent information to $\Delta$ and is, in a certain sense, its *canonical Alexander dual*:

$$\Delta^* := \{\, F \subseteq [n] : [n] - F \notin \Delta \,\}.$$

The crucial property of $\Delta^*$ that makes it convenient for the study of $\mathrm{Tor}^A_{\cdot}(R, K)$ is that, instead of the full subcomplexes of $\Delta$ that are relevant in Hochster's formula, the relevant subcomplexes of $\Delta^*$ are the *links* of its faces. Therefore, various hypotheses on $\Delta^*$ which are inherited by the links of faces, or which control the topology of these links, lead to strong consequences for $\mathrm{Tor}^A_{\cdot}(R, K)$ (see Section 3).

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