Configurations of Linear Subspaces and Rational Invariants

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1. Introduction

Let $\mathrm{Gr}_{n,d}(\mathbb{C})$ denote the Grassmannian of all d-dimensional linear subspaces in $\mathbb{C}^n$, and let $\mathrm{GL}_n(\mathbb{C}) \times (\mathrm{Gr}_{n,d}(\mathbb{C}))^s \to (\mathrm{Gr}_{n,d}(\mathbb{C}))^s$ be the canonical diagonal action. Dolgachev [DB] posed the following question:

> *Is the quotient* $\mathrm{Gr}_{n,2}(\mathbb{C})^s/\mathrm{GL}_n(\mathbb{C})$ (*e.g., in the sense of Rosenlicht*) *always rational?*

Recall that a Rosenlicht quotient of an algebraic variety X acted on by an algebraic group G is an algebraic variety V together with a rational map $X \to V$ whose generic fibers coincide with the G-orbits. Such quotients always exist and are unique up to birational isomorphisms [R]. In the sequel all quotients will be assumed of this type. An algebraic variety Q is *rational* if it is birationally equivalent to $\mathbb{P}^m$ with $m = \dim Q$.

We answer the above question in the affirmative by applying the rationality of the quotient $(\mathrm{GL}_2(\mathbb{C}))^2/\mathrm{GL}_2(\mathbb{C})$, where $\mathrm{GL}_2(\mathbb{C})$ acts diagonally by conjugations (see [P1] and surveys [B; D]).

THEOREM 1.1. *For all positive integers n and s, the quotient* $(\mathrm{Gr}_{n,2}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ *is rational. Equivalently, the field of rational* $\mathrm{GL}_n(\mathbb{C})$*-invariants on* $(\mathrm{Gr}_{n,2}(\mathbb{C}))^s$ *is pure transcendental.*

The statement of Theorem 1.1 has been recently proved by Megyesi [M] in the case $n = 4$ and by Dolgachev and Boden [DB] in the case of odd n. Their proofs are independent of the present one.

More generally, we show the birational equivalence between $(\mathrm{Gr}_{n,d}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ and certain quotients of matrix spaces. Let $\mathrm{GL}_n(\mathbb{C}) \times (\mathrm{GL}_n(\mathbb{C}))^s \to (\mathrm{GL}_n(\mathbb{C}))^s$ be the action defined by $(g, M_1, \dots, M_s) \mapsto (gM_1g^{-1}, \dots, gM_sg^{-1})$. The first main result of the present paper consists of the following two statements.

THEOREM 1.2.

(1) *Let s and d be arbitrary positive integers, and let* $n = rd$ *for some integer* $r > 1$. *Then* $(\mathrm{Gr}_{n,d}(\mathbb{C}))^s/\mathrm{GL}_n(\mathbb{C})$ *is birationally equivalent to* $(\mathrm{GL}_d(\mathbb{C}))^k/\mathrm{GL}_d(\mathbb{C})$, *where*

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