# Homology of Real Algebraic Fiber Bundles Having Circle as Fiber or Base

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## 1. Introduction

For real algebraic sets $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$, a map $F: X \to Y$ is said to be *entire rational* if there exist $f_i, g_i \in \mathbb{R}[x_1, \dots, x_r]$, $i = 1, \dots, s$, such that each $g_i$ vanishes nowhere on $X$ and $F = (f_1/g_1, \dots, f_s/g_s)$. We say $X$ and $Y$ are *isomorphic* to each other if there are entire rational maps $F: X \to Y$ and $G: Y \to X$ such that $F \circ G = \mathrm{id}_Y$ and $G \circ F = \mathrm{id}_X$. A *complexification* $X_{\mathbb{C}} \subseteq \mathbb{CP}^N$ of $X$ will mean that $X$ is a nonsingular algebraic subset of some $\mathbb{RP}^N$ and $X_{\mathbb{C}} \subseteq \mathbb{CP}^N$ is the complexification of the pair $X \subseteq \mathbb{RP}^N$. We also require the complexification to be nonsingular (blow up $X_{\mathbb{C}}$ along smooth centers away from $X$ defined over reals if necessary). For basic definitions and facts about real algebraic geometry, we refer the reader to [2; 4]. Let $KH_*(X, R)$ be the kernel of the induced map

$$i_*: H_*(X, R) \to H_*(X_{\mathbb{C}}, R)$$

on homology, where $i: X \to X_{\mathbb{C}}$ is the inclusion map and $R$ is either $\mathbb{Z}$ or a field. In [16] it is shown that $KH_*(X, R)$ is independent of the complexification $X \subseteq X_{\mathbb{C}}$. All compact manifolds and nonsingular real or complex algebraic sets are $R$ oriented so that Poincaré duality and intersection of homology classes are defined.

In this note, $X$ will be mostly the total space of a fiber bundle and we will study $KH_*(X, R)$. In the next section the fiber will be $S^1$ and in the third section the base space will be $S^1$. As an application we will prove a result of Kulkarni that a compact homogeneous manifold $M$ has an algebraic model $X$ with $[X]$ zero in $H_n(X_{\mathbb{C}}; \mathbb{Z})$ if and only if $M$ has zero Euler characteristic. (Kulkarni [10, Cor. 4.6, Thm. 5.1] proved this for rational coefficients.) In Section 4 we will consider entire rational maps $f: X \to Y$ and compare $KH_k(X, R)$ and $KH_k(Y, R)$ via $f$ in case $X$ and $Y$ have the same dimension. Results will be proved in the last section.

## 2. Bundles with Circle Fibers

On any compact Lie group there is a unique real algebraic structure compatible with the group operations [12]. Let $G$ be such a group endowed with its unique real algebraic structure. An action of $G$ on $X$ is said to be *algebraic* if the action

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