The Grunsky Operator and the Schatten Ideals

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1. Introduction

Let J be a bounded Jordan curve in the complex plane. It divides the Riemann sphere into two simply connected Jordan domains Ω, Ω^* with Riemann maps g, g_* from the unit disc U and the exterior of the closed disc U^*, which extend as homeomorphisms of the boundary. We study the Grunsky operator Γ_g (defined in the next section) and its relationship to the welding homeomorphism $h = g_*^{-1} \circ g$ of the unit circle to itself for certain classes of smooth quasicircles. We recall two theorems (which are equivalent).

Theorem 1.1 (Pommerenke [7]). *Let g be a conformal map of the unit disc to a simply connected region Ω. Then $\partial\Omega$ is a quasicircle if and only if the Grunsky operator Γ_g, acting on the Dirichlet space, has norm less than* 1.

Theorem 1.2 (Beurling and Ahlfors [3]). *Let J be a Jordan curve in the plane with welding h. Then J is a quasicircle if and only if the composition operator $V_h\colon f(z) \to f(h(z))$ is bounded on the Dirichlet space.*

These theorems are related by the idea of a conformal map acting as a composition operator. We will sketch this in Section 2, and in the remainder of the paper will prove the following two theorems.

Theorem 1.3. *Let g be a conformal map of the unit disc to the interior of a Jordan curve. The Grunsky operator lies in the p*th *Schatten ideal γ_p $(p \geq 1)$ of operators on the Dirichlet space if and only if $\log g' \in B_p$, the Besov space.*

Theorem 1.4. *Let J be a quasicircle with welding h. The commutator $[V_h, H]$ of V_h with the Hilbert transform H lies in γ_p if and only if $\log g' \in B_p$, where g is the conformal map to the interior.*

The proofs of the theorems will be straightforward applications of atomic decompositions of Bergman spaces and quasiconformal estimates, given our initial descriptions of the welding and the Grunsky operator. We note the obvious analogy

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