The Algebra of Unbounded Continuous Functions on a Stonean Space and Unbounded Operators

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1. Introduction

The investigation of commutative operator algebras by means of function space techniques is due to M. H. Stone [7]. The notion of a space such that the closure of every open set G, $\mathrm{clos}(G)$, is open (thus, $\mathrm{clos}(G)$ is *clopen*) was introduced by Stone in [8]. Such spaces are called *extremely disconnected*. Extremely disconnected spaces are also characterized as those topological spaces X for which (i) the interior of a closed subset F of X, $\mathrm{int}(F)$, is clopen, or (ii) disjoint open subsets of X have disjoint closures. A compact Hausdorff extremely disconnected space X is also known as a *Stonean space.* If $\mathcal{A}$ is an abelian von Neumann algebra then $\mathcal{A}$ is isomorphic with $C(X)$, where X is a Stonean space (see [5, Thm. 5.2.1]).

In [4] (and [5]), Kadison studies a class of unbounded continuous complex-valued (real-valued) functions on an extremely disconnected space X (called *normal functions* and *self-adjoint functions* and denoted by $N(X)$ and $S(X)$, respectively), and he proves that $N(X)$ is an algebra [4, Thm. 2.11]. Starting with an abelian von Neumann algebra $\mathcal{A}$, Kadison introduces $N(\mathcal{A})$, the algebra of (normal) operators affiliated with $\mathcal{A}$ and $S(\mathcal{A})$, the algebra of self-adjoint operators affiliated with $\mathcal{A}$ [4, Thm. 3.3], extending the isomorphism of $\mathcal{A}$ with $C(X)$ to a *-isomoprhism of $N(\mathcal{A})$ onto $N(X)$ [4, Thm. 4.1]. In this direction, one is enabled to obtain the spectral theorem for self-adjoint and normal operators (see also [2]).

In this article, we present a closely related approach to the study of $N(X)$, $S(X)$, and the spectral theorem for unbounded self-adjoint operators. We begin in Section 2 with a theorem (Theorem 2.1) on continuous extensions from open dense subsets of extremely disconnected spaces (see also [3, p. 96]). Theorem 2.1 leads to a substantial simplification of the proof that $N(X)$ is an algebra, and it plays a key role in our development. We continue, in Section 3, with a discussion on the spectral analysis of a function in $S(X)$, and we give an alternative proof of the fact that $S(X)$ is a boundedly complete lattice. In Section 4 we prove the spectral theorem and characterizations of the spectrum and the spectral projections for unbounded self-adjoint operators.

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