Uniform Estimates for the Hyperbolic Metric and Euclidean Distance to the Boundary

Barbara B. Flinn & David A. Herron

Dedicated to Alan F. Beardon, for his interest and insightful discussions

1. Introduction

Throughout this article, D is a proper subdomain of the complex plane $\mathbb{C}$ possessing at least two finite boundary points, usually termed a *hyperbolic* domain. Each such D carries constant negative curvature metrics, and we let λ_D denote the scale factor or density for the maximal constant curvature -1 metric. We call λ_D the *Poincaré hyperbolic metric* for D; it can be defined by

$$\lambda_D(z) = \lambda_{\mathbb{B}}(\zeta)/|p'(\zeta)| = 2/(1-|\zeta|^2)|p'(\zeta)|,$$

where $z = p(\zeta)$ and $p\colon \mathbb{B} \to D$ is any holomorphic covering projection from the unit disk $\mathbb{B} = \{|\zeta| < 1\}$ onto D. See [BP; HM; M1; M2] and their references for basic properties of the Poincaré metric.

An elementary exercise using Schwarz's lemma shows that λ_D satisfies a domain monotonicity property, from which we easily conclude that

$$\lambda_D(z)\operatorname{dist}(z, \partial D) \le 2 \tag{1.1}$$

for all points $z \in D$ for any hyperbolic domain D. In the opposite direction, an application of Koebe's one-quarter theorem [P3, 1.4, p. 9] yields

$$\lambda_D(z)\operatorname{dist}(z, \partial D) \ge 1/2 \tag{1.2}$$

for all $z \in D$ when D is simply connected. Thus we see from (1.1) and (1.2) that, in simply connected hyperbolic domains D, the Poincaré metric and the Euclidean distance to the boundary ∂D of D are approximately reciprocals; however, for general hyperbolic domains there are no universal lower bounds as in (1.2).

It is well known that equality holds in (1.1) (resp., (1.2)) at some point z if and only if D is a disk centered at z (resp., D is the complement of a ray and z lies on the ray of symmetry). Our purpose here is to investigate when strict inequality holds *uniformly* in (1.1) or (1.2). We exhibit geometric conditions that provide estimates for the quantities

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