# Relatively Hyperbolic Groups

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## 1. Introduction

Let $V$ be a complete, noncompact Riemannian manifold of constant negative curvature with finite volume. Then $V$ has only finitely many ends, $E_1, E_2, \dots, E_k$. The inclusions $E_i \subset V$ induce injections $\pi_1(E_i) \hookrightarrow \pi_1(V)$ $(i = 1, 2, \dots, k)$. The relations between the fundamental group $\pi_1(V)$ and the subgroups $\pi_1(E_i)$ are the main motivation for introducing a theory of hyperbolic groups relatively to the family of subgroups (for short, relatively hyperbolic groups). The idea is similar to the case of the fundamental group of compact hyperbolic manifold. Its geometric and combinatorial structure gives us the definition of word-hyperbolic groups [1; 3; 7].

There are two definitions of relatively hyperbolic groups. The first one proposed in [7] by Gromov (cf. Definition 1) is a generalization of the parabolic properties of the subgroups $\pi_1(E_i)$. The second definition (cf. Definition 2), proposed by Farb in [4] (see also [5]), is expressed by properties of the modification of the Cayley graph (coned-off Cayley graph) of $\pi_1(V)$. According to the first definition, it is obvious that $\pi_1(V)$ is hyperbolic relatively to the family of subgroups $\pi_1(E_i)$ $(i = 1, 2, \dots, k)$. Farb's definition is weaker, and the proof of the hyperbolicity of $\pi_1(V)$ relatively to the family $\pi(E_i)$ $(1 \le i \le k)$ becomes more difficult. However, it is convenient for constructing many illustrative examples. In this note we want to prove (Theorem 1) that the Gromov definition is stronger than the one by Farb, and we give an example (Example 3) of a group that is relatively hyperbolic in the sense of Farb's definition but is *not* relatively hyperbolic in the sense of Gromov's definition.

The paper is organized as follows. In Section 2 we formulate the two definitions of relatively hyperbolic groups and give some examples. This part is based on [7, 8.6] and [4, 1.1]. In Section 3 we prove our main result (Theorem 1) that the Farb definition is more general than the Gromov definition. The main idea of proof, which was proposed to us by Brian Bowditch, is the following proposition.

Proposition. *Let $(X, d)$ be a $\delta$-hyperbolic metric space $(\delta \ge 0)$ with the collection of closed disjoint $\varepsilon$-quasiconvex subsets. Let each subset contract to a point.*

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