Nonexistence of Some Quasi-Conformal Harmonic Diffeomorphisms

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1. Introduction

The property of harmonic maps between complete Riemannian manifolds has been studied extensively by many authors (e.g., [C; Sh; T]). In the present paper we show some nonexistence results for quasi-conformal harmonic diffeomorphisms between complete Riemannian manifolds. In dimension 2, harmonic maps are closely related to the deformation theory of Riemann surfaces. One of the questions that arises naturally is: *Are Riemann surfaces that are related by harmonic diffeomorphisms necessarily quasi-conformally related?* See Schoen's article [S] for a general discussion on this subject, where other questions were also discussed. The result we show in this paper provides some partial answers to the high-dimensional generalization of this type of question. In particular we prove the following result, which can be thought of as a Liouville type theorem for harmonic diffeomorphisms.

THEOREM 1.1. *Let M^n be a complete manifold with $\mathrm{Ricci}_M \geq 0$, and let N^n be a simply connected manifold with nonpositive sectional curvature, where n is the dimension of both manifolds. If there is a point $p \in M$ such that $\lim_{r\to\infty} V_p(r) = o(r^n)$, then there is no quasi-conformal harmonic diffeomorphism from M into N with polynomial growth energy density.*

It is not surprising that the growth rate of energy density plays a role here. For example, Wan proved [W] that a harmonic diffeomorphism between hyperbolic spaces of dimension 2 is quasi-conformal if and only if it has bounded energy density. The "only if" part of Wan's theorem was generalized to high dimension in [LTW]. Where it was proved that if the Ricci curvature of the domain manifold is bounded from below and if the first eigenvalue of the target manifold is positive, then any quasi-conformal harmonic diffeomorphism into the target manifold has bounded energy density. These results and some other related results in [HTTW] all indicate that the growth condition on the energy density is a natural assumption and is closely related to the study of quasi-conformal diffeomorphisms. On the other hand, we can show by examples that Theorem 1.1 will not be

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