Immiscible Fluid Clusters in $\mathbf{R}^2$ and $\mathbf{R}^3$

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1. Introduction

Immiscible fluids $F_1, \ldots, F_m$ in $\mathbf{R}^n$ (with ambient F_0) have an energy proportional to surface area, where the constant of proportionality $a_{ij} > 0$ depends on which fluids the surface separates. To prevent degeneracies, we assume strict triangle inequalities $a_{ik} < a_{ij} + a_{jk}$. B.White [W1; W4, Sec. 11] has announced that energy-minimizing clusters of prescribed volumes are smooth surfaces of constant mean curvature that meet along a singular set of Hausdorff dimension at most $n-2$. In $\mathbf{R}^2$ it would follow that an energy-minimizing cluster consists of arcs of circles meeting at isolated points. Our Regularity Theorem 4.3 gives a simple proof special to $\mathbf{R}^2$.

The special case of planar soap-bubble clusters ($a_{ij} = 1$) was treated in [M2]. That simple analysis generalizes immediately to the case of $m = 2$ immiscible fluids and to the case where each $a_{ij} \approx 1$.

1.1. The Proof of Regularity Theorem 4.3. Proposition 4.2 shows that if C is, in a small ball $\mathbf{B}(a, r)$, weakly close to a diameter separating (say) fluid F_1 from $F_0 = 0$, then C is a circular arc in a shrunken ball $\mathbf{B}(a, 0.9r)$. Its proof first uses projection in the space of coefficients to replace C inside $\mathbf{B}(a, r)$ with a cluster C' whose coefficients are all real multiples of F_1. It follows from the strict triangle inequality that this reduces cost at least $\varepsilon|C - C'|$. A circular arc is even cheaper. Second, lost amounts of other fluids, on the order of $|C - C'|^2$ by the isoperimetric inequality, may be restored elsewhere at cost $K|C - C'|^2$. At a small scale, $K|C - C'|^2 < \varepsilon|C - C'|$ and the circular arc is better.

To deduce Regularity Theorem 4.3, note that in a small ball about any point, C is weakly close to a tangent cone, which must consist of finitely many rays. By the previous Proposition 4.2, C consists of nearly radial circular arcs that meet at the point.

1.2. Clusters in $\mathbf{R}^3$. Section 5 generalizes Taylor's classification of soap-bubble cluster singularities to clusters of immiscible fluids with interface energies near unity.

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