Injective Operations of Homogeneous Spaces

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1. Introduction

This paper is an extension of Conner and Raymond's work [2]. A torus T^k can be viewed as a homogeneous space $\mathbb{R}^k/\mathbb{Z}^k$. Let G be a simply connected divisible Lie group, and let Γ be a co-compact discrete subgroup of G such that (Γ, G) has the unique automorphism extension property. Even if G/Γ is not a group, there is a natural concept of an "action" of the homogeneous space G/Γ in place of a torus, which gives rise to useful facts generalizing known results of torus actions.

There have been many efforts trying to split a manifold as a product of two manifolds. Let M be a flat Riemannian manifold whose fundamental group contains a nontrivial center. Calabi has shown that such an M almost splits. More precisely, there exists a compact flat manifold N and a finite abelian group Φ such that $M = T^k \times_\Phi N$, the quotient space of $T^k \times N$ by a free diagonal action of Φ, where Φ acts freely as translations on the first factor and as isometries on the second factor (see [17]). Lawson and Yau [9] and Eberlein [4] have shown the same fact for closed manifolds M of nonpositive sectional curvature: If $\pi_1(M)$ has nontrivial center $\mathbb{Z}^k$ then M splits as $M = T^k \times_\Phi N$, where N is a closed manifold of nonpositive sectional curvature and Φ is a finite abelian group acting diagonally and freely on T^k-factors as translations.

Prior to the work described in the previous paragraph, Conner and Raymond [2] generalized Calabi's results to homologically injective torus actions. Let (T^k, M) be a torus action on a topological space. For a base point $x_0 \in M$, consider the evaluation map $\mathrm{ev}\colon (T^k, e) \to (M, x_0)$ sending $t \mapsto tx_0$. The action is called *injective* if the evaluation map induces an injective homomorphism $\mathrm{ev}_\#\colon \pi_1(T^k, e) \to \pi_1(M, x_0)$. It is *homologically injective* if the evaluation map induces an injective homomorphism $\mathrm{ev}_*\colon H_1(T^k, \mathbb{Z}) \to H_1(M; \mathbb{Z})$. For a Riemannian manifold of nonpositive sectional curvature, the existence of a nontrivial center $\mathbb{Z}^k$ of $\pi_1(M)$ guarantees that the manifold has an action of torus T^k; and all such actions are homologically injective.

Topological spaces are always assumed to be paracompact, path-connected, locally path-connected, and either (i) locally compact and semi-1-connected or

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