Holomorphic Motions of Hyperbolic Sets

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0. Introduction

Let M be a complex Hermitian manifold and $\{f_a\}_{a\in\mathbf{D}}$ a holomorphic family of endomorphisms of M, where $\mathbf{D}$ is the unit disk. This means that the map $\mathbf{D}\times M\to M$, defined by $(a,x)\to f_a(x)$, is holomorphic. Suppose that $f=f_0$ has a compact surjectively invariant subset K, that is, $f(K)=K$. For example, K could be a fixed point or a periodic orbit, but also a more complicated set such as the Julia set of a rational function. We may then ask if K is persistent under the perturbation f_a of the map f. For instance, if K is a fixed point of f, then we ask if f_a has a fixed point K_a near K for a small enough. A sufficient (albeit not necessary) condition for this is that the fixed point K be hyperbolic, meaning that the derivative of f at K has no eigenvalue of modulus 1.

There is a natural notion of hyperbolicity for general sets K. Let us first consider the case when the maps f_a are diffeomorphisms. The precise definition (which can be found e.g. in [R]) will not be stated here, but it says that the tangent bundle over K splits continuously into two invariant subbundles on which the derivative of f is expanding and contracting, respectively.

One basic result in real dynamics is that hyperbolic sets are persistent under perturbations in the map f (see [R]). In our case this means that if a is small enough, then f_a has a hyperbolic set K_a close to K, and there exists a homeomorphism h_a close to the identity conjugating $f|_K$ to $f_a|_{K_a}$.

If K is a hyperbolic fixed point, then it follows from the implicit function theorem that the fixed point K_a of f_a depends holomorphically on a. The natural generalization of this to more general sets K is the notion of a holomorphic motion, the definition of which is given in Section 1.

Theorem A. *Let $\{f_a\}_{a\in\mathbf{D}}$ be a holomorphic family of diffeomorphisms of a Hermitian manifold M parameterized by the unit disk $\mathbf{D}$. Suppose that $f=f_0$ has a hyperbolic subset K. Then K moves holomorphically with the parameter a at $a=0$. More precisely, there exist $r>0$ and a holomorphic motion $h\colon \mathbf{D}_r\times K\to M$ such that, for each $a\in\mathbf{D}_r$:*

(1) *$K_a:=h(a,K)$ is a hyperbolic subset for f_a;*

(2) *the map $h_a:=h(a,\cdot)\colon K\to K_a$ is a homeomorphism and $f_a\circ h_a=h_a\circ f$.*

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