Holomorphic Sections of Prequantum Line Bundles on G/N

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1. Introduction

Let K be a compact connected semisimple Lie group, let G be its complexification, and let $G = KAN$ be an Iwasawa decomposition. Let T be the centralizer of A in K, so that $H = TA$ is a Cartan subgroup of G. Since G and N are complex, G/N is a complex manifold. Besides the left G-action on G/N, there is also a right H-action because H normalizes N.

In [10], Schwarz suggests the following scheme of geometric quantization on the space G/N: Equip G/N with a K-invariant Kähler structure ω, and consider the corresponding prequantum line bundle $\mathbf{L}$ [6; 9]. Namely, the Chern class of $\mathbf{L}$ is the cohomology class $[\omega]$, and $\mathbf{L}$ has a connection ∇ whose curvature is ω. In fact, we shall see that if ω is Kähler then it is exact, so $\mathbf{L}$ is just a trivial bundle. However, the geometry arising from the connection is interesting. Given a section s of $\mathbf{L}$, we say that s is holomorphic if $\nabla_\xi s = 0$ for every antiholomorphic vector field ξ. Let $H(\mathbf{L})$ denote the holomorphic sections of $\mathbf{L}$. The K-action on G/N lifts to a K-representation on $H(\mathbf{L})$. Let $\mathfrak{k}$ be the Lie algebra of K. Then the infinitesimal representation on $H(\mathbf{L})$ is given by

$$\xi \cdot s = \nabla_{\xi^\sharp} s + \sqrt{-1}\phi^\xi s, \quad \xi \in \mathfrak{k},\ s \in H(\mathbf{L}) \tag{1.1}$$

[6, (3.1)], where $\xi^\sharp$ is the infinitesimal vector field on G/N induced by the left K-action and $\xi \mapsto \phi^\xi$ is the moment map $\mathfrak{k} \longrightarrow C^\infty(G/N)$ corresponding to the K-action preserving ω. Note that the moment map exists, since K is semisimple [7]. A K-invariant Kähler structure on G/N has potential function if and only if it is invariant under the right T-action [3]. In joint work with Guillemin [4], we carry out the foregoing construction for such Kähler structures and prove the following theorem.

Theorem. *Let ω be a K-invariant Kähler structure on G/N. If it is right T-invariant, then $H(\mathbf{L})$ contains every finite-dimensional irreducible K-representation with multiplicity* 1.

Such a representation is called a *model* if it is equipped with a unitary structure—a term due to Gelfand and Zelevinski [5]. The preceding theorem is an analog of

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