Weighted L^2-Cohomology of Bounded Domains with Smooth Compact Quotients

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1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. The Bergman metric on Ω is a Kähler metric invariant under the group $\operatorname{Aut}(\Omega)$ of biholomorphic automorphisms of Ω. Denote the Bergman metric on Ω by ds^2_Ω, and denote its Kähler form by ω. For $0 \le p, q \le n$ we denote by $\mathcal{H}_2^{p,q}(\Omega)$ the space of square integrable harmonic (p,q)-forms on Ω with respect to ds^2_Ω. When the boundary of Ω is smooth, Donnelly and Fefferman proved the following result.

Theorem [DF]. *If Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$, then*

$$\dim \mathcal{H}_2^{p,q}(\Omega) = \begin{cases} 0 & \text{if } p+q \neq n, \\ \infty & \text{if } p+q = n. \end{cases} \tag{1.1}$$

See also [D], where Donnelly gave an alternative proof of this theorem using a criterion of Gromov [Gro].

It is known that (1.1) also holds for bounded symmetric domains whose boundaries are not smooth in general (see [Gro] and [Ka]). It is thus natural to ask: Does (1.1) hold for bounded domains in C^n without any conditions on the boundary? An important class of bounded domains are those that cover compact manifolds, and they have been extensively studied (see e.g. [Ca; Fr; Kob; Si; V]). In this article, we consider the spaces of harmonic forms on such domains that are square integrable with respect to certain weight functions. Our result can be regarded as a partial affirmative answer to the above question for such domains.

For $z \in \Omega$, we denote by $d(z) = \operatorname{dist}(z; \partial\Omega)$ the Euclidean distance between z and the boundary $\partial\Omega$ of Ω. For $s \in \mathbb{R}$ we define

$$\mathcal{H}_{2,s}^{p,q}(\Omega) := \left\{ \phi \in \mathcal{A}^{p,q}(\Omega) \;\middle|\; \Box\phi = 0 \text{ and } \int_\Omega \|\phi(z)\|^2 \frac{1}{d(z)^s} \frac{\omega^n}{n!} < \infty \right\}. \tag{1.2}$$

Here $\Box$ and $\|\cdot\|$ denote (respectively) the Laplacian and the pointwise norm with respect to ds^2_Ω. It is easy to see that, for $s > 0$, each $\mathcal{H}_{2,s}^{p,q}(\Omega)$ forms a vector subspace of $\mathcal{H}_2^{p,q}(\Omega)$ and

$$\mathcal{H}_{2,s}^{p,q}(\Omega) \subset \mathcal{H}_{2,s'}^{p,q}(\Omega) \quad \text{if } s \ge s'. \tag{1.3}$$

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