Higher-Dimensional Analogs of Hermite's Constant

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Introduction

For integers $n > 1$, Hermite's constant is the smallest number γ_n such that, for all lattices $\Lambda \subset \mathbb{R}^n$ of rank n, there is a nonzero lattice point $\mathbf{x} \in \Lambda$ with

$$\|\mathbf{x}\| \leq \gamma_n^{1/2} \det(\Lambda)^{1/n}.$$

Here $\|\mathbf{x}\|$ denotes the usual Euclidean length of $\mathbf{x}$. Hermite was the first to prove the existence of such a constant. He showed that

$$\gamma_n \leq \gamma_{n-1}^{(n-1)/(n-2)} \tag{1}$$

for $n > 2$. Using (1) and a quick induction argument gives $\gamma_n \leq \gamma_2^{n-1}$. After verifying that $\gamma_2 = 2/\sqrt{3}$, Hermite arrived at the upper bound $\gamma_n \leq (2/\sqrt{3})^{n-1}$. Later, Minkowski used his first convex bodies theorem (see [3]) to show that

$$\gamma_n \leq 4V(n)^{-2/n}, \tag{2}$$

where $V(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. Note that this upper bound for γ_n grows linearly in n as $n \to \infty$, as opposed to the exponential growth of Hermite's original upper bound.

Note that, by introducing a scaling factor, we may restrict to lattices of determinant 1 in the definition of Hermite's constant (see Lemma 4). Minkowski's work on the space of such lattices led him to state (without proof) that

$$\gamma_n \geq \left(\frac{2\zeta(n)}{V(n)}\right)^{2/n}. \tag{3}$$

This result was first proven by Hlawka (see [3, Sec. 19]); it is a special case of what is now called the Minkowski–Hlawka theorem. This, along with Minkowski's upper bound stated in (2), shows that γ_n in fact grows linearly in n as $n \to \infty$. It is not known whether γ_n/n approaches a limit as $n \to \infty$. The exact value of γ_n is known only for $n \leq 8$ (see [3]).

Hermite's constant is directly related to the densest lattice packing of spheres in $\mathbb{R}^n$, and through this to many areas of mathematics and even other natural sciences (number theory, lie algebras, numerical integration, chemistry, and digital

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