Quasiconformally Homogeneous Compacta in the Complex Plane

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To F. W. Gehring on the occasion of his 70th birthday

1. Introduction

This paper summarizes some of the findings of the co-authors in their investigation of sets E in the extended complex plane $\hat{\mathbb{C}}$ that exhibit a high degree of symmetry with respect to the action of the quasiconformal group. Our focus is exclusively on compact subsets E of $\hat{\mathbb{C}}$. A situation roughly dual to this one, the case of quasiconformally homogeneous domains, was studied in [GP] and [Sa].

We begin by establishing some convenient notation and terminology. The symbol $\mathcal{T}$ stands for the group of sense-preserving homeomorphisms of $\hat{\mathbb{C}}$ to itself; $\mathcal{Q}$ signifies the subgroup of $\mathcal{T}$ that comprises all the quasiconformal self-mappings of $\hat{\mathbb{C}}$; for $1 \le K < \infty$, $\mathcal{Q}_K$ denotes the family of mappings in $\mathcal{Q}$ that are K-quasiconformal. (N.B. We observe accepted convention in requiring as part of the definition of a plane quasiconformal mapping that it be sense-preserving, although orientation will not be a serious concern in what follows.) The family $\mathcal{Q}_1$, be it noted, is nothing other than the classical Möbius group, the group of linear fractional transformations of $\hat{\mathbb{C}}$. By contrast, when $K > 1$ the family $\mathcal{Q}_K$ is not closed under composition and so does not constitute a group. For each nonempty subset E of $\hat{\mathbb{C}}$ we write

$$\mathcal{T}(E) = \{\, f \in \mathcal{T} : f(E) = E \,\}, \quad \mathcal{Q}(E) = \mathcal{Q} \cap \mathcal{T}(E), \quad \mathcal{Q}_K(E) = \mathcal{Q}_K \cap \mathcal{T}(E).$$

Thus $\mathcal{T} = \mathcal{T}(\hat{\mathbb{C}})$, $\mathcal{Q} = \mathcal{Q}(\hat{\mathbb{C}})$ and $\mathcal{Q}_K = \mathcal{Q}_K(\hat{\mathbb{C}})$.

A nonempty subset E of $\hat{\mathbb{C}}$ is said to be *quasiconformally homogeneous* (resp., *K-quasiconformally homogeneous*) if the action on E of the group $\mathcal{Q}(E)$ (resp., the family $\mathcal{Q}_K(E)$) is transitive: for each pair of points a and b of E there exists a mapping f in $\mathcal{Q}(E)$ (resp., in $\mathcal{Q}_K(E)$) such that $f(a) = b$. Since $\mathcal{Q}_K(E)$ is not a group when $K > 1$, this definition does entail a slight departure from the standard meaning of "action." The expression "conformally homogeneous" will be employed as a preferred synonym for "1-quasiconformally homogeneous." We define the *index of quasiconformal homogeneity* $\mathcal{K}(E)$ of a quasiconformally homogeneous set E in $\hat{\mathbb{C}}$ by

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