Inner Functions in the Hyperbolic Little Bloch Class

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1. Introduction

The hyperbolic derivative of an analytic self-map $\varphi: D \to D$ of the unit disk is given by $|\varphi'|/(1-|\varphi|^2)$. To explain the terminology, we note that integrating $|\varphi'|/(1-|\varphi|^2)$ over a rectifiable curve γ in D gives the hyperbolic arclength of $\varphi(\gamma)$. This notion of derivative has been used by Yamashita to study hyperbolic versions of the classical Hardy and Dirichlet spaces; see [Y1] and [Y2]. More recently, in [MM] and [SZ], hyperbolic derivatives have been shown to be pertinent to the study of composition operators on certain subspaces of $H(D)$, the space of analytic functions on D. An analytic self-map φ of D induces a linear operator $C_\varphi: H(D) \to H(D)$ defined by $C_\varphi f = f \circ \varphi$. This operator is called the *composition operator* induced by φ.

Recall that an analytic function f on D is said to belong to the Bloch space $\mathcal{B}$ provided that $(1-|z|^2)|f'(z)|$ is uniformly bounded for $z \in D$. Similarly, $f \in \mathcal{B}_0$, the little Bloch space, if $(1-|z|^2)|f'(z)| \to 0$ uniformly as $|z| \to 1$. The hyperbolic Bloch class $\mathcal{B}^h$ is defined by using the hyperbolic derivative in place of the ordinary derivative in the definition of the Bloch space. That is, $\varphi \in \mathcal{B}^h$ if $\varphi: D \to D$ is analytic and

$$\sup_{z\in D} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} < \infty.$$

Similarly, we say $\varphi \in \mathcal{B}_0^h$, the hyperbolic little Bloch class, if $\varphi \in \mathcal{B}^h$ and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} = 0.$$

Note that these are not linear spaces, since φ is required to be a self-map of D. It is an easy consequence of the Schwarz–Pick lemma that every analytic self-map of D belongs to $\mathcal{B}^h$, and in fact the supremum above is at most 1; see [G, p. 2]. Membership in the hyperbolic little Bloch class, on the other hand, is nontrivial.

It is easy to see that $C_\varphi: \mathcal{B} \to \mathcal{B}$ is bounded for every analytic self-map φ of D, while $C_\varphi: \mathcal{B}_0 \to \mathcal{B}_0$ is bounded if and only if $\varphi \in B_0$. It is a recent result of Madigan and Matheson that $C_\varphi: \mathcal{B}_0 \to \mathcal{B}_0$ is compact if and only if $\varphi \in \mathcal{B}_0^h$;

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